Polynomial and Continued-Fraction Approximations for Logarithmic Functions

By Kurt Spielberg

1. Introduction. In this article we present the coefficients of approximations which are well suited for the calculation of logarithms on digital computers. The approximations have been derived by means of the IBM 704 program IB CTR. They are chosen so as to approximately minimize the absolute error over the appropriate interval of the argument. The method is described in detail in references [1], [2].

Similar selected polynomial approximations have been made available by C. Hastings [3]. The approximations of the present article, however, cover a much wider range of accuracy and should allow the coding of efficient double-precision subroutines.

Continued fraction approximations have been used systematically by E. G. Kogbetliantz and the author in connection with subroutines for the IBM 704 and 709 computers (see e.g. [4], [5], [6], which contain many references to other literature on rational approximations). The reader should note that the continued fraction approximations given in this paper not only allow for computation with fewer second-order arithmetic operations (multiplications and divisions) but also are intrinsically more accurate than polynomial approximations with equal numbers of constants.

2. Polynomial Approximations. In the case of digital computers, the argument can be assumed to be in normalized floating point form:

A. Binary machines:

$$y = 2^i \cdot f \tag{1.a}$$

$i \cdots$ integer, $f \cdots$ fraction, $(\frac{1}{2}) \leqq f < 1$.

B. Decimal machines:

$$y = 10^I \cdot F \tag{1.b}$$

$I \cdots$ integer, $F \cdots$ fraction, $(\frac{1}{10}) \leqq F < 1$.

The natural logarithm is then evaluated in accordance with the relations:

$$\log_e y = (i + \log_2 f) \cdot \log_e 2 \tag{2.a}$$

$$\log_e y = (I + \log_{10} F) \cdot \log_e 10. \tag{2.b}$$

To obtain efficient polynomial approximations, one starts with the well known series

$$\log_e \frac{v + x}{v - x} = 2[(x/v) + (x^3/3v^3) + (x^5/5v^5) + \cdots] \tag{3}$$

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which converges in the interval $[-v < x < v]$. Since we intend to "economize" the power series by means of Chebyshev polynomials, we focus our attention on the interval $[-1 \leqq x \leqq 1]$. To this end we introduce the rational transformations

$$(4.a) \qquad f = u \cdot \left(\frac{v + x}{v - x}\right), \qquad x = v \cdot \left(\frac{f - u}{f + u}\right)$$

$$(4.b) \qquad F = U \cdot \left(\frac{V + x}{V - x}\right), \qquad x = V \cdot \left(\frac{F - U}{F + U}\right)$$

and determine u, v, U, V so that the interval $[-1 \leqq x \leqq 1]$ maps one-to-one onto the intervals $[(\frac{1}{2}) \leqq f \leqq 1]$ and $[(\frac{1}{10}) \leqq F \leqq 1]$ respectively.

The parameters are determined from the endpoint conditions:

$$(5.a) \qquad u \cdot \left(\frac{v + 1}{v - 1}\right) = 1, \; u \cdot \left(\frac{v - 1}{v + 1}\right) = \frac{1}{2} \qquad \rightarrow u = 1/\sqrt{2}, \quad v = (\sqrt{2} + 1)^2$$

$$(5.b) \qquad U \cdot \left(\frac{V + 1}{V - 1}\right) = 1, \; U \cdot \left(\frac{V - 1}{V + 1}\right) = \frac{1}{10} \qquad \rightarrow U = 1/\sqrt{10}, \quad V = (\sqrt{10} + 1)^2/9.$$

On substituting these values into equation 3, we obtain the following power series for $\log_2 f$ and $\log_{10} F$:

$$(6.a) \qquad \log_2 f = 2 \cdot \log_2 e \cdot [(x/v) + (x^3/3v^3) + \cdots] - (\tfrac{1}{2})$$

$$(6.b) \qquad \log_{10} F = 2 \cdot \log_{10} e \cdot [(x/V) + (x^3/3V^3) + \cdots] - (\tfrac{1}{2}), \qquad [-1 \leqq x \leqq 1]$$

The 704 program IB CTR is now applied to produce polynomial approximations to the functions $\log_2 f + (\frac{1}{2})$ and $\log_{10} F + (\frac{1}{2})$. These approximations have the form:

$$(7) \qquad f_m^*(x) = \sum_{i=1}^{m} \bar{c}_{2i-1}^{(m)} \cdot x^{2i-1}$$

$$x = v \frac{f - (\sqrt{2}/2)}{f + (\sqrt{2}/2)} \quad \text{for} \quad \log_2 f + \left(\frac{1}{2}\right)$$

$$x = V \frac{F - (\sqrt{10}/10)}{F + (\sqrt{10}/10)} \quad \text{for} \quad \log_{10} F + \left(\frac{1}{2}\right).$$

For computational purposes, however, it is preferable to introduce the variables $z = \dfrac{x}{v}$ or $z = \dfrac{x}{V}$,

$$(8) \qquad f_m^*(z) = \sum_{i=1}^{m} c_{2i-1}^{(m)} \cdot z^{2i-1}.$$

In Tables 1 and 2 we give the coefficients $c_j^{(m)}$ for those m which result in approximations of less than or equal to 16-digit accuracy. IB CTR performs operations to 16-digit accuracy only. Its primary output, however, consists of the increments Δa_i which have to be added to the power series coefficients a_i of the given function to produce the coefficients of the approximation polynomial. We therefore give

TABLE 1
Polynomial Coefficients for $(Log_2 f + \frac{1}{2})$
[Format: 451 E-16 = (.451) × 10^{-16}]

# of coefficients = 2, E1 = 606 E-05,	E2 = 560 E-05, E3 = 876 E-04			
2885 2290 8878 0725 E01	9835 1099 0073 3683 E00			
# of coefficients = 3, E1 = 317 E-07,	E2 = 299 E-07, E3 = 185 E-05			
2885 3912 8434 1961 E01	9614 7149 2111 3942 E00	5989 5531 8769 7430 E00		
# of coefficients = 4, E1 = 182 E-09,	E2 = 174 E-09, E3 = 423 E-07			
2885 3900 7279 5173 E01	9618 0074 7520 7173 E00	5765 8533 5684 3571 E00	4342 4052 2333 9827 E00	
# of coefficients = 5, E1 = 111 E-11,	E2 = 106 E-11, E3 = 102 E-08			
2885 3900 8184 5024 E01	9617 9664 8473 7566 E00	5770 8662 4639 5350 E00	4115 3509 8458 0017 E00	3428 0712 2893 2386 E00
# of coefficients = 6, E1 = 699 E-14,	E2 = 680 E-14, E3 = 254 E-10			
2885 3900 8177 7425 E01	9617 9669 4401 5285 E00	5770 7788 7562 8954 E00	4122 1350 6252 1815 E00	3197 6122 8252 9226 E00
2846 8436 5237 6334 E00				
# of coefficients = 7, E1 = 451 E-16,	E2 = 111 E-15, E3 = 649 E-12			
2885 3900 8177 7930 E01	9617 9669 3921 2376 E00	5770 7801 8088 9661 E00	4121 9830 3814 7730 E00	3206 2195 4010 7357 E00
2612 9289 6508 8097 E00	2444 9890 0547 1920 E00			

TABLE 1′
Increments to Polynomial Coefficients for $(Log_2 f + \frac{1}{2})$

# of coefficients = 8, E1 = 296 E-18,	E3 = 168647 E-13			
−2796 4430 3552 4398 E-16	4551 4828 0666 6136 E-13	−2159 4856 0182 7467 E-10	4596 5458 7348 9827 E-08	−5181 9115 5104 8872 E-06
3306 6195 2511 6075 E-04	−1196 4114 9893 8173 E-02	2271 7162 0774 9647 E-01		
# of coefficients = 9, E1 = 197 E-20,	E3 = 444326 E-15			
2088 8319 0383 6615 E-18	−4250 6312 1996 8916 E-15	2536 1559 2048 5863 E-12	−6874 1931 2702 1155 E-10	1008 3697 9885 3810 E-07
−8676 6985 8599 0271 E-06	4501 4264 0509 3858 E-04	−1381 0846 5746 4859 E-02	2289 0231 2019 8052 E-01	
# of coefficients = 10, E1 = 133 E-22,	E3 = 118370 E-16			
−1560 2706 2751 6011 E-20	3881 2970 6574 1623 E-17	−2842 8056 6023 4796 E-14	9543 6054 6158 3398 E-12	−1759 9844 9576 6640 E-09
1949 1463 1860 0759 E-07	−1350 9195 6439 6997 E-05	5896 2316 8612 9454 E-04	−1569 3718 5748 8830 E-02	2306 5044 3068 2277 E-01
# of coefficients = 11, E1 = 906 E-25,	E3 = 318215 E-18			
1165 4539 6029 6268 E-22	−3479 5084 3296 5673 E-19	3068 2604 2428 8200 E-16	−1248 3015 3206 5234 E-13	2820 0813 2649 2660 E-11
−3889 3504 0329 2187 E-09	3441 0313 3886 3375 E-07	−1992 1520 3031 4775 E-05	7497 3891 8807 8451 E-04	−1761 3336 7553 1971 E-02
2324 1621 0740 7857 E-01				
# of coefficients = 12, E1 = 622 E-27,	E3 = 861954 E-20			
−8705 4029 5087 7987 E-25	3071 9811 7588 6013 E-21	−3209 5080 2494 8594 E-18	1554 8328 4036 6693 E-15	−4216 4739 5337 7175 E-13
7064 8242 3847 3655 E-11	−7727 2510 5334 2210 E-09	5674 1027 1296 3337 E-07	−2816 6347 8303 9504 E-05	9311 4020 8001 2885 E-04
−1957 0316 5775 8419 E-02	2341 9981 4599 0346 E-01			

Table 2

Polynomial Coefficients for $(Log_{10} F + \frac{1}{2})$

# of coefficients = 2, E1 = 687 E-03,	E2 = 635 E-03, E3 = 818 E-02			
8632 4020 7770 2620 E00	3631 7894 3815 3434 E00			
# of coefficients = 3, E1 = 380 E-04,	E2 = 358 E-04, E3 = 161 E-02			
8690 0869 5758 3972 E00	2776 7991 9776 1202 E00	2534 4884 5752 8303 E00		
# of coefficients = 4, E1 = 230 E-05,	E2 = 219 E-05, E3 = 342 E-03			
8685 5602 6257 9886 E00	2910 9861 8255 4689 E00	1540 0462 1701 3371 E00	2105 6233 5032 9898 E00	
# of coefficients = 5, E1 = 147 E-06,	E2 = 141 E-06, E3 = 763 E-04			
8685 9154 8508 7103 E00	2893 4361 3145 7315 E00	1774 1562 6792 7102 E00	9489 8398 4169 7810 E-01	1904 8229 9261 9342 E00
# of coefficients = 6, E1 = 971 E-08,	E2 = 938 E-08, E3 = 175 E-04			
8685 8876 0977 6415 E00	2895 5019 3437 1026 E00	1731 2899 5934 1548 E00	1312 0429 7994 7967 E00	5595 3156 3065 2789 E-01
1812 6871 4549 2145 E00				
# of coefficients = 7, E1 = 659 E-09,	E2 = 639 E-09, E3 = 412 E-05			
8685 8897 9723 0560 E00	2895 2749 8124 8346 E00	1738 0176 3929 2238 E00	1226 5721 9388 9737 E00	1087 3756 0172 2274 E00
2479 7381 8779 1450 E-01	1783 9825 1660 6317 E00			
# of coefficients = 8, E1 = 455 E-10,	E2 = 443 E-10, E3 = 986 E-06			
8685 8896 2557 4872 E00	2895 2987 2744 9401 E00	1737 0673 4730 0434 E00	1243 3400 2720 4158 E00	9354 9710 3959 7025 E-01
9846 9859 1034 3788 E-01	−3594 1212 1911 1870 E-02	1798 2956 4764 1127 E00		
# of coefficients = 9, E1 = 319 E-11,	E2 = 311 E-11, E3 = 239 E-06			
8685 8896 3904 5177 E00	2895 2963 3160 9180 E00	1737 1916 3411 5875 E00	1240 4452 2516 2623 E00	9712 5210 4496 9738 E-01
7341 7616 3051 0283 E-01	9637 6057 5788 3042 E-01	−3184 6422 2742 1276 E-01	1845 5330 9203 7982 E00	
# of coefficients = 10, E1 = 226 E-12,	E2 = 221 E-12, E3 = 585 E-07			
8685 8896 3798 8125 E00	2895 2965 6661 9593 E00	1737 1763 0775 8544 E00	1240 8995 5170 2479 E00	9639 5703 5930 4128 E-01
8029 8336 5029 0451 E-01	5715 1350 0078 7728 E-01	1010 4006 5135 6441 E00	−6164 7796 8076 9668 E-01	1920 5939 8039 2521 E00
# of coefficients = 11, E1 = 162 E-13,	E2 = 159 E-13, E3 = 144 E-07			
8685 8896 3807 1075 E00	2895 2965 4407 9458 E00	1737 1781 1181 2187 E00	1240 8333 4031 9255 E00	9652 9278 6613 0788 E-01
7867 8488 0187 9711 E-01	6946 3643 1539 0860 E-01	4194 8195 6636 4441 E-01	1122 3305 1516 4517 E00	−9429 7998 7036 6938 E-01
2021 1022 0368 1399 E00				
# of coefficients = 12, E1 = 117 E-14,	E2 = 117 E-14, E3 = 359 E-08			
8685 8896 3806 4564 E00	2895 2965 4620 2003 E00	1737 1779 0732 3385 E00	1240 8424 3295 8296 E00	9650 6817 2073 5434 E-01
7901 7458 8805 0730 E-01	6617 8630 0558 8470 E-01	6281 5170 1450 0886 E-01	2581 5225 2642 8773 E-01	1304 1471 6289 5274 E00
−1309 5310 3342 3318 E00	2146 3381 2708 5184 E00			

\# of coefficients = 13, E1 = 847 E-16, E2 = 111 E-15, E3 = 900 E-09

8685 8896 3806 5076 E00	2895 2965 4600 5157 E00	1737 1779 2978 8923 E00	1240 8412 4374 9968 E00	9651 0342 4344 7121 E-01
7895 2858 8846 5882 E-01	6695 1982 7477 7399 E-01	5659 2707 1451 3706 E-01	5972 2476 4507 6175 E-01	6977 8384 6561 5379 E-02
1565 4748 5242 4707 E00	−1727 5334 3561 2921 E00	2296 7005 4645 3325 E00		

\# of coefficients = 14, E1 = 662 E-17, E2 = 555 E-16, E3 = 226 E-09

8685 8896 3806 5035 E00	2895 2965 4602 3178 E00	1737 1779 2738 4921 E00	1240 8413 9306 2811 E00	9650 9819 9026 3539 E-01
7896 4264 9078 3005 E-01	6678 7260 4454 3685 E-01	5822 0354 4727 4206 E-01	4854 7126 9236 7508 E-01	6025 3255 6326 7152 E-01
−1642 0035 6283 8796 E-01	1920 3899 2012 4037 E00	−2209 0134 7726 2696 E00	2473 4265 8441 2308 E00	

Table 2′
Increments to Polynomial Coefficients for ($Log_{10} F + \frac{1}{2}$)

# of coefficients = 13, E1 = 847 E-16, E2 = 111 E-15, E3 = 899533 E-09				
4008 4239 3911 5484 E-14	−1802 1583 5441 1154 E-11	2404 0008 8731 8954 E-09	−1493 1284 3346 2257 E-07	5225 3183 5780 7491 E-06
−1140 6023 1713 0290 E-04	1647 2230 2337 1189 E-03	−1627 6473 2760 5324 E-02	1117 5349 5270 8717 E-01	−5327 5417 1670 5883 E-01
1729 6752 0870 8676 E00	−3647 9233 5573 7152 E00	4505 7140 2371 7118 E00		
# of coefficients = 14, E1 = 662 E-17, E2 = 555 E-16, E3 = 226368 E-09				
−2916 4626 3720 0172 E-15	1497 3893 0252 3668 E-12	−2280 8840 3338 6883 E-10	1619 6375 3240 7678 E-08	−6496 4754 0810 2044 E-07
1631 8345 1136 6819 E-05	−2727 5231 9866 8126 E-04	3144 2355 2308 3100 E-03	−2546 3415 3553 1193 E-02	1453 8047 0112 7640 E-01
−5778 1414 8572 6928 E-01	1542 7425 4455 5990 E00	−2556 4490 6278 5299 E00	2151 7269 6818 7675 E00	
# of coefficients = 15, E1 = 486 E-18, E3 = 572275 E-10				
2288 5535 9684 9383 E-16	−1343 8391 4610 9992 E-13	2344 3219 5244 5695 E-11	−1911 7146 2297 6515 E-09	8843 4268 4050 7464 E-08
−2577 4243 7703 7674 E-06	5040 3255 9895 0395 E-05	−6875 4658 7637 3031 E-04	6691 8269 8604 0228 E-03	−4693 0443 9505 2550 E-02
2365 8076 4831 7931 E-01	−8427 8096 1729 4303 E-01	2040 0398 5490 6148 E00	−3088 8853 0687 4070 E00	2378 9344 2405 9968 E00
# of coefficients = 16, E1 = 358 E-19 E3 = 145258 E-10				
−1795 9736 6410 3296 E-17	1195 8805 9185 7760 E-14	−2368 5279 2028 3146 E-12	2197 7461 1744 1895 E-10	−1160 8259 9300 6863 E-08
3881 8413 3500 7086 E-07	−8767 5766 2421 1050 E-06	1393 5293 3967 7936 E-04	−1599 0567 3726 0520 E-03	1343 4272 4682 2011 E-02
−8295 8548 8882 7127 E-02	3740 1090 5421 8566 E-01	−1206 2550 3494 5501 E00	2668 5531 8199 7598 E00	−3716 4406 6478 1781 E00
2634 1345 0733 5488 E00				
# of coefficients = 17, E1 = 265 E-20, E3 = 370012 E-11				
1409 3150 8547 3774 E-18	−1056 3115 8573 2524 E-15	2357 1276 8150 9970 E-13	−2468 8104 8591 3208 E-11	1476 1050 4700 8316 E-09
−5609 8840 3888 0974 E-08	1447 6875 5677 5826 E-06	−2647 4854 3802 6277 E-05	3527 7025 8319 1047 E-04	−3483 5465 5112 1759 E-03
2570 0434 3901 5269 E-02	−1416 0855 4267 2644 E-01	5769 6629 1498 4858 E-01	−1698 8701 6376 2035 E00	3458 3781 1427 7836 E00
−4455 4183 0129 9020 E00	2921 0185 6128 9769 E00			
# of coefficients = 18, E1 = 196 E-21, E3 = 945511 E-12				
−1105 9688 8198 7785 E-19	9268 7827 8567 9632 E-17	−2314 5895 1668 1069 E-14	2717 1493 4957 1495 E-12	−1825 1852 6724 5195 E-10
7818 7429 4464 7917 E-09	−2284 2814 5794 9697 E-07	4756 2823 4455 0482 E-06	−7268 9080 6071 2682 E-05	8311 1740 8052 1467 E-04
−7188 5605 8617 9683 E-03	4721 4631 9690 9278 E-02	−2346 5322 9941 6829 E-01	8715 5992 3026 6765 E-01	−2359 5803 5285 6509 E00
4445 9765 4492 1987 E00	−5324 8659 4849 2503 E00	3243 7845 5215 9573 E00		
# of coefficients = 19, E1 = 146 E-22, E3 = 242297 E-12				
8678 6815 8826 4611 E-21	−8085 0793 9863 8296 E-18	2245 8153 1814 8675 E-15	−2936 4513 5611 5064 E-13	2201 3687 5694 3754 E-11
−1055 3520 0829 8306 E-09	3463 0894 7261 7532 E-08	−8137 0689 8363 0670 E-07	1411 7100 4334 4387 E-05	−1846 3409 3646 6263 E-04
1844 5616 7118 6270 E-03	−1417 2101 0408 5692 E-02	8379 7119 6535 3521 E-02	−3790 3518 8690 6554 E-01	1292 8323 5304 1614 E00
−3237 7162 4057 8437 E00	5675 4563 7087 2527 E00	−6347 0335 4087 0700 E00	3607 2084 8706 5067 E00	
# of coefficients = 20, E1 = 109 E-23, E3 = 622496 E-13				
−6810 2228 2251 4817 E-22	7015 1294 5830 8251 E-19	−2155 8115 2234 0397 E-16	3121 9959 4939 8246 E-14	−2596 6481 6481 3679 E-12
1384 3409 4148 1999 E-10	−5067 1511 6295 6044 E-09	1333 2738 1946 9607 E-07	−2603 0804 1617 8951 E-06	3855 0774 2268 1785 E-05

−4395 2520 2266 6984 E-04	3892 3608 8149 6460 E-03	−2687 1533 6395 1063 E-02	1443 6598 9658 1256 E-01	−5988 2739 0797 6818 E-01
1887 4523 9004 8467 E00	−4395 5168 5808 9775 E00	7200 0996 3927 2206 E00	−7547 9067 2022 7777 E00	4016 7260 9602 3941 E00

# of coefficients = 21, E1 = 816 E-25, E3 = 160299 E-13

5344 3892 5310 2525 E-23	−6057 5655 6179 9451 E-20	2049 4387 4450 8078 E-17	−3270 6768 7331 8013 E-15	3002 1538 1713 7558 E-13
−1769 8835 7932 5375 E-11	7182 5272 3571 8276 E-10	−2102 2162 7827 6335 E-08	4584 3885 2468 5963 E-07	−7622 3540 8301 8407 E-06
9819 0989 1950 0592 E-05	−9903 7750 4168 7679 E-04	7866 3901 2604 2533 E-03	−4926 2758 2856 7780 E-02	2423 5452 9496 9185 E-01
−9278 5253 8561 7140 E-01	2717 1618 6520 7351 E00	−5911 2413 0888 5684 E00	9084 1861 8185 0081 E00	−8957 8287 4109 7084 E00
4478 5262 7490 0559 E00				

# of coefficients = 22, E1 = 612 E-26, E3 = 413655 E-14

−4193 8089 2404 0763 E-24	5207 9021 4801 2185 E-21	−1931 2330 2346 2839 E-18	3380 9297 1741 7593 E-16	−3408 6150 5610 5388 E-14
2210 9742 3128 4624 E-12	−9894 1765 6093 8458 E-11	3202 3242 6442 6305 E-09	−7749 4621 9111 2255 E-08	1435 9844 9702 4511 E-06
−2072 5304 1843 1378 E-05	2357 4164 4914 3829 E-04	−2128 8798 5630 2663 E-03	1531 4262 9655 0943 E-02	−8769 5043 5061 7099 E-02
3976 9690 6299 1939 E-01	−1413 1624 4153 8475 E00	3863 1553 0659 5049 E00	−7882 9871 1669 8231 E00	1140 5168 5158 9211 E01
−1061 2225 3706 5133 E01	4999 6580 0646 9901 E00			

# of coefficients = 23, E1 = 457 E-27, E3 = 106950 E-14

3291 1012 0129 4949 E-25	−4459 5968 1784 8354 E-22	1805 2889 8664 4529 E-19	−3452 5888 7283 9814 E-17	3806 8023 8461 8771 E-15
−2704 5103 1627 6891 E-13	1328 1452 5263 4629 E-11	−4728 7227 5666 3148 E-10	1262 5826 3924 1371 E-08	−2590 7962 4956 1663 E-07
4159 2197 4371 8105 E-06	−5290 9424 1824 6419 E-05	5379 3390 1236 8934 E-04	−4392 8399 1546 7543 E-03	2885 4594 6430 3172 E-02
−1521 2097 0908 0803 E-01	6395 8604 1552 0965 E-01	−2119 6352 6547 6698 E00	5431 6316 5652 5391 E00	−1043 3368 7038 7462 E01
1425 6264 1187 6378 E01	−1255 2449 7724 7013 E01	5588 1527 2783 5117 E00		

# of coefficients = 24, E1 = 320 E-28, E3 = 277007 E-15

−2582 5814 8673 8534 E-26	3804 8849 0235 4618 E-23	−1675 1940 2563 1511 E-20	3486 6979 1696 7982 E-18	−4187 9243 2519 6063 E-16
3245 3634 1605 9773 E-14	−1741 3525 5568 6211 E-12	6788 3390 1234 3116 E-11	−1989 6513 4147 8110 E-09	4495 7986 2581 7089 E-08
−7977 8302 5787 6452 E-07	1126 9152 1233 6463 E-05	−1279 3277 9037 5553 E-04	1174 4587 7238 1308 E-03	−8745 5383 9347 7507 E-03
5282 1375 3009 4918 E-02	−2578 8727 3323 3044 E-01	1010 2742 2496 2924 E00	−3136 0204 9913 4927 E00	7560 8121 7592 3551 E00
−1371 5241 6034 4965 E01	1774 9543 3483 0017 E01	−1482 6767 1200 5955 E01	6253 1643 9083 9764 E00	

tables of these increments from which the reader can construct approximations of great accuracy by simple hand computation.

All approximations of the form (7) have been tested at more than 100 points in the interval $[-1 \leqq x \leqq 1]$. Instead of the complete error curves we submit, for simplicity, three "error parameters."

$E_1 \cdots$ a theoretical upper bound of the magnitude of the absolute error caused by a truncation of a Chebyshev series to m terms

$E_2 \cdots$ the maximum magnitude of the absolute error encountered in the described test

$E_3 \cdots \sum_{i=m+1}^{\infty} a_{2i-1}$, the maximum absolute error incurred by a truncation of the given power series to m terms.

The sets of increments have been tested as follows. From the definitions we infer that (for $x = 1$)

$$\sum_{i=1}^{m} a_{2i-1} + \sum_{i=m+1}^{\infty} a_{2i-1} = \sum_{i=1}^{m} (a_{2i-1} + \Delta a_{2i-1}) \pm \max (E_1, E_2)$$

or

$$E_3 = \sum_{i=1}^{m} \Delta a_{2i-1} \pm \max (E_1, E_2).$$

Selected tests of this type have consistently been satisfactory. The reader should note, however, that these tests do not usually apply to the last two digits due to the unfortunate fact that E_3 has been printed only to 6 digits. In order to obtain a better check, at least up to "triple precision accuracy" on the IBM 704 (2^{-70}), we have therefore coded a triple precision logarithm subroutine based on the given increments. The accuracy of the subroutine was verified by an application to functional relationships of the form $\log (x \cdot y) = \log x + \log y$. We have every reason to believe that all of the given increments will be found to be completely accurate.

3. Continued Fraction Approximations. An approximation polynomial can be transformed into a rational approximation with the same number of constants by means of the "multiple truncation procedure" described in [2] and implemented in IB CTR. It is shown in [2] that the rational approximation may actually be considerably better than the original polynomial approximation. The results submitted in the present article furnish an excellent instance of this behavior.

Rational approximations can readily be transformed into continued fractions which can be evaluated in fewer operations. In Tables 3 and 4 we give the continued fraction expressions for $(\log_2 f + \frac{1}{2})$ and $(\log_{10} F + \frac{1}{2})$ up to 16-digit accuracy. They are of the form

$$g_m^*(z)/z = H_0 + \frac{G_1 |}{| z^2 + H_1} + \frac{G_2 |}{| z^2 + H_2} + \cdots + \frac{G_{[m/2]} |}{| z^2 + H_{[m/2]}},$$

where $m = 3, 4, \cdots$ and $[\frac{1}{2}m]$ is the largest integer $\leqq \frac{m}{2}$. For even m, the constant H_0 is zero.

Table 3

Continued Fraction Coefficients for ($Log_2\ F + \frac{1}{2}$)

# of coefficients = 3, E1 = 482 E-08,	E2 = 479 E-08, E3 = 185 E-05		
1292 0070 9870 0440 E01	−1656 7626 3013 4752 E01		
−2639 8577 0311 1530 E01			
# of coefficients = 4, E1 = 719 E-11,	E2 = 971 E-11, E3 = 423 E-07		
0000 0000 0000 0000 00	−7987 3541 1394 9024 E01	−1893 0810 4263 2588 E01	
−1747 9113 9907 0586 E02	−3652 8036 5468 0985 E01		
# of coefficients = 5, E1 = 577 E-14,	E2 = 226 E-13, E3 = 102 E-08		
8270 7235 6521 4108 E00	−3085 7167 7071 2198 E01	−1540 1793 1703 5943 E01	
−5536 8085 7347 3842 E01	−6095 2425 1953 1276 E00		
# of coefficients = 6, E1 = 794 E-16,	E2 = 500 E-15, E3 = 254 E-10		
0000 0000 0000 0000 00	−1561 2646 7876 6829 E02	−3704 0518 2875 6762 E01	−1390 3458 4589 7226 E01
−2641 9662 6270 7515 E02	−2281 0512 9374 2221 E02	−2377 2595 4404 4343 E00	
# of coefficients = 7, E1 = 100 E-17,	E2 = 167 E-15, E3 = 649 E-12		
6089 0779 0129 1574 E00	−5026 9809 0578 5215 E01	−2550 5195 2771 6355 E01	−1306 2235 3597 6247 E01
−8465 8394 7460 5167 E01	−3212 8370 0542 7867 E01	−1234 5730 6822 9921 E00	

Constants are listed in the following sequence:

First line (or lines): $H_0, H_1, \cdots, H_{[m/2]}$

New line: $G_1, G_2, \cdots, G_{[m/2]}$

TABLE 4

Continued Fraction Coefficients for $(Log_{10} F + \frac{1}{2})$

# of coefficients = 3, E1 = 271 E-05,	E2 = 587 E-05, E3 = 161 E-02			
4174 8684 7078 8742 E00	−1567 8846 4453 5199 E01			
−7073 8930 5767 5742 E00				
# of coefficients = 4, E1 = 314 E-06,	E2 = 184 E-06, E3 = 342 E-03			
0000 0000 0000 0000 00	−7006 1359 6128 1824 E01	−1741 4420 1618 7053 E01		
−4806 7940 1892 9581 E01	−2563 5634 1028 4778 E01			
# of coefficients = 5, E1 = 434 E-07,	E2 = 877 E-08, E3 = 763 E-04			
2735 8316 6302 7741 E00	−2733 5680 1437 2866 E01	−1431 6382 7957 1771 E01		
−1460 7439 0859 0961 E01	−3987 9825 1331 3200 E00			
# of coefficients = 6, E1 = 497 E-08,	E2 = 744 E-09, E3 = 175 E-04			
0000 0000 0000 0000 00	−1217 4701 9298 6194 E02	−2958 5851 1583 7019 E01	−1269 2331 5593 5871 E01	
−6828 6213 0481 5603 E01	−1234 6312 3127 7424 E02	−1218 1864 1232 5462 E00		
# of coefficients = 7, E1 = 605 E-09,	E2 = 589 E-10, E3 = 412 E-05			
2257 0476 5469 1600 E00	−3630 0459 1199 2254 E01	−1860 6441 8950 3682 E01	−1081 7612 3251 9165 E01	
−1928 2949 9018 7036 E01	−1158 3549 7192 2684 E01	−2568 2365 4014 4450 E-01		
# of coefficients = 8, E1 = 119 E-09,	E2 = 481 E-11, E3 = 986 E-06			
0000 0000 0000 0000 00	−1552 9064 0908 9411 E02	−3714 7223 4968 7198 E01	−1412 1175 9204 4839 E01	−7168 7986 8486 6398 E00
−7920 3098 1162 8075 E01	−2267 8348 7376 9289 E02	−2498 4529 6288 2307 E00	−5063 2909 8671 8532 E-03	
# of coefficients = 9, E1 = 156 E-10,	E2 = 623 E-12, E3 = 239 E-06			
1925 4014 9937 8824 E00	−4664 6742 0574 2049 E01	−2390 9878 0081 7684 E01	−1274 9458 0829 0809 E01	−3522 2956 7225 2972 E00
−2393 3428 1057 6800 E01	−2600 4549 6660 8233 E01	−9996 4987 5811 9110 E-01	−3358 7524 7535 1047 E-07	
# of coefficients = 10, E1 = 434 E-12,	E2 = 156 E-11, E3 = 585 E-07			
0000 0000 0000 0000 00	−2278 8796 3815 2385 E02	−5544 3240 6207 0304 E01	−1934 3441 8577 6231 E01	−1218 2064 2048 6353 E01
−1897 0504 6669 6781 E00	0000 0000 0000 0000 00	0000 0000 0000 0000 00	0000 0000 0000 0000 00	0000 0000 0000 0000 00
−9900 4585 3391 4664 E01	−5795 9838 5747 5431 E02	−8607 6714 8406 6327 E00	−5625 5676 2020 1057 E-01	−7655 7921 1811 2189 E-12
# of coefficients = 11, E1 = 254 E-13,	E2 = 603 E-13, E3 = 144 E-07			
1566 4660 0628 8101 E00	−6592 2236 1492 1441 E01	−3455 8641 8980 6945 E01	−1697 0477 1482 5547 E01	−1180 8990 8952 5397 E01
−1369 5502 7639 1718 E00	0000 0000 0000 0000 00	0000 0000 0000 0000 00	0000 0000 0000 0000 00	0000 0000 0000 0000 00
−3126 5747 7180 0882 E01	−7074 7971 1034 8246 E01	−4016 2950 6095 0182 E00	−3557 5429 4422 7386 E-01	−1782 4702 3079 4942 E-12
# of coefficients = 12, E1 = 187 E-14,	E2 = 334 E-13, E3 = 359 E-08			
0000 0000 0000 0000 00	−3169 9223 8947 9199 E02	−7831 1526 9137 3795 E01	−2568 4206 7268 4757 E01	−1525 7373 3977 9744 E01
−1144 0138 7296 9406 E01	−1201 4681 1859 7815 E00	0000 0000 0000 0000 00	0000 0000 0000 0000 00	0000 0000 0000 0000 00

−1193 3840 8458 0869 E02	−1251 0738 8678 6443 E03	−2104 3075 1900 4511 E01	−2053 9093 1711 4612 E00	−2160 1176 0348 0808 E-01
−5688 4767 1640 4652 E-12				
# of coefficients = 13, E1 = 174 E-15, E2 = 149 E-13, E3 = 900 E-09				
1393 1454 3501 9793 E00	−8062 3888 3507 8176 E01	−4276 3916 4675 1179 E01	−2007 5909 6834 5058 E01	−1343 0501 1722 4303 E01
−1061 3732 5582 5836 E01	−1130 8536 1647 6820 E00	0000 0000 0000 0000 00	0000 0000 0000 0000 00	0000 0000 0000 0000 00
−3615 1063 3147 0115 E01	−1206 7169 2646 9193 E02	−7573 1788 7233 3270 E00	−9019 7116 9776 3227 E-01	−7211 9042 7185 6326 E-02
−2655 8838 7827 0656 E-11				
# of coefficients = 14, E1 = 867 E-17, E2 = 153 E-13, E3 = 226 E-09				
0000 0000 0000 0000 00	−3693 2105 0974 0172 E02	−9153 8599 6300 3995 E01	−2900 6086 3002 2923 E01	−1656 2571 8132 1678 E01
−1191 9233 8543 8025 E01	−7551 8137 2851 1484 E00	−1068 6185 0036 1872 E00	0000 0000 0000 0000 00	0000 0000 0000 0000 00
−1299 9443 2325 4275 E02	−1766 0355 0332 9192 E03	−3020 2074 0796 5379 E01	−3080 7913 8786 6877 E00	−3556 1104 2194 9375 E-01
−6747 9929 9079 2036 E-04	−2239 5413 9687 9253 E-09			

All continued fractions have been checked at more than 100 points in the interval $[-1 \leqq x \leqq 1]$. These checks were executed in double precision arithmetic, i.e., with wordlengths of 16 digits. The user of a particular continued fraction approximation should briefly analyze how much round-off error may accrue on his machine due to limited wordlength and subtraction of numbers of equal magnitude. For large H_i and G_i serious loss of accuracy might occur in this manner. In the case of the logarithmic functions, however, little difficulty should arise from this source.

4. Use of Tables. To illustrate the use of the tables we give a few simple examples.

a) Polynomial approximation, three coefficients:

$$\frac{1}{2} \leqq f \leqq 1, \qquad z = \frac{f - \frac{\sqrt{2}}{2}}{f + \frac{\sqrt{2}}{2}}$$

$$\log_2 f = f_3^*(z) - \tfrac{1}{2} \pm (.32)\ .10^{-7} = c_1 z + c_3 z^3 + c_5 z^5 - \tfrac{1}{2} \pm (.32)\ .10^{-7}$$

Table 1:

$$c_1 = 2.88539\quad 12843 \cdots$$

$$c_3 = .96147\quad 14921 \cdots$$

$$c_5 = .59895\quad 53187 \cdots$$

Another way of writing the approximation would be

$$\frac{1}{\sqrt{2}} \leqq x \leqq \sqrt{2}, \qquad z = \frac{x-1}{x+1}$$

$$\log_2 x = c_1 z + c_3 z^3 + c_5 z^5 \pm (.32)\ .10^{-7}.$$

b) Continued fraction approximation, three coefficients:

$$\frac{1}{2} \leqq f \leqq 1, \qquad z = \frac{f - \frac{\sqrt{2}}{2}}{f + \frac{\sqrt{2}}{2}}$$

$$\log_2 f = g_3^*(z) - \frac{1}{2} \pm (.48)\ .10^{-8} = z\left[H_0 + \frac{G_1}{z^2 + H_1}\right] - \frac{1}{2} \pm (.48)\ .10^{-8}$$

Table 3:

$$H_0 = 1.29200\quad 70987$$

$$H_1 = -1.65676\quad 26301$$

$$G_1 = -2.63985\quad 77031.$$

c) Use of increments:

The increments of Tables 1′ and 2′ should be added to the coefficients:

$$(2 \log_2 e,\ \tfrac{2}{3} \log_2 e,\ \tfrac{2}{5} \log_2 e,\ \cdots)$$

and

$$(2 \log_{10} e,\ \tfrac{2}{3} \log_{10} e,\ \tfrac{2}{5} \log_{10} e,\ \cdots)$$

respectively.

In our computations we have used the constants

$$2 \log_2 e = 2.88539\ 00817\ 77926\ 8146$$

$$2 \log_{10} e = .86858\ 89638\ 06503\ 6553.$$

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