also includes approximate values of $\int_{-1}^{1} Z^2 dx$, suitably normalized with respect to the value of Z or dZ/dx at $x = 0$ and at $x = 1$, for $\eta = 0.0001$ and for representative values of $\eta^3 n^2$ in the range 0 through 20. This report is available from the Office of Technical Services, U. S. Department of Commerce. Two copies of the report have been deposited in the file of Unpublished Mathematical Tables which is maintained by *Mathematics of Computation* and may be made available on loan to interested individuals.

Department of Physics and Institute for Atomic Research
Iowa State University
Ames, Iowa

E. I. du Pont de Nemours & Company
Wilmington, Delaware

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On the Computation of Lommel's Functions of Two Variables

By J. Boersma

In 1942 Zernike and Nijboer [1], [2] introduced a new expansion of Lommel's functions of two variables in connection with calculating the diffraction integral of a circular aperture. In this article it is shown that this expansion is very well suited for the computation of these functions. (The author is much indebted to Dr. Bottema of the Physical Laboratory of the University of Groningen, who drew his attention to this formula.)

Lommel's functions of two variables are defined in the following way (Cf. [3],

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formulas 16.5 (5) and (6), p. 537, 538),

$$(1)\qquad \begin{aligned} U_\nu(w, z) &= \sum_{m=0}^{\infty} (-1)^m \left(\frac{w}{z}\right)^{\nu+2m} J_{\nu+2m}(z) \\ V_\nu(w, z) &= \cos\left(\frac{w}{2} + \frac{z^2}{2w} + \frac{\nu\pi}{2}\right) + U_{-\nu+2}(w, z). \end{aligned}$$

The present article deals with the computation of Lommel's functions of two variables of integral order. Owing to the recurrence formulas, (Cf. [3], formulas 16.5 (7) and (8), p. 538),

$$(2)\qquad \begin{aligned} U_\nu(w, z) + U_{\nu+2}(w, z) &= \left(\frac{w}{z}\right)^{\nu} J_\nu(z) \\ V_\nu(w, z) + V_{\nu+2}(w, z) &= \left(\frac{w}{z}\right)^{-\nu} J_{-\nu}(z), \end{aligned}$$

it is sufficient to compute Lommel's functions for two successive integral values of ν.

The first table of Lommel's functions of two variables of integral order is to be found in Lommel's memoir on diffraction at a circular aperture [4]. Lommel gives tables for $\frac{2}{w}U_1(w, z)$, $\frac{2}{w} U_2(w, z)$, and for $\frac{2}{w} V_0(w, z)$, $\frac{2}{w} V_1(w, z)$ to six decimal places for values of the arguments $w = \pi(\pi)10\pi$, $z = 0(1)12$, and $w = \pi(\pi)8\pi$, $z = 0(1)12$ respectively.

Quite recently, a table [5] by Dekanosidze has been published which gives tables of $U_1(w, z)$, $U_2(w, z)$, $V_1(w, z)$, $V_2(w, z)$ to six decimal places for a somewhat uncommon domain of values of the arguments:

$$w = 0.5(0.02)1.2(0.05)4(0.1)6.2, \qquad z = w(0.01)4\sqrt{w}$$

$$w = 6.3(0.1)10, \qquad z = w(0.01)10.$$

The tables may also be used outside this domain of values by means of the relations (Cf. [5], formulas (7) and (8))

$$(3)\qquad \begin{aligned} U_n(w, z) &= (-1)^n V_n\left(\frac{z^2}{w}, z\right) \\ V_n(w, z) &= (-1)^n U_n\left(\frac{z^2}{w}, z\right). \end{aligned}$$

Dekanosidze's tables have been computed by means of a power series expansion of $U_\nu(w, z)$ and $V_\nu(w, z)$ in powers of $\frac{z^2}{2w}$ (Cf. [5], formula (3)),

$$(4)\qquad \begin{aligned} U_\nu(w, z) &= \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!}\left(\frac{z^2}{2w}\right)^m U_{\nu+m}(w, 0) \\ V_\nu(w, z) &= \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!}\left(\frac{z^2}{2w}\right)^m V_{\nu-m}(w, 0). \end{aligned}$$

When ν is integral, the coefficients of these power series contain a factor of the type $U_n(w, 0)$ and $V_n(w, 0)$, where n is an integer. $U_n(w, 0)$ and $V_n(w, 0)$ are given by [5], formula (4), which contains some printing errors. (Cf. [3], formulas 16.52 (11)–(16), p. 540). The correct formulas are as follows:

$$
(5)\qquad
\begin{aligned}
U_{2n}(w,0) &= (-1)^n \left[\cos\frac{w}{2} - \sum_{m=0}^{n-1} (-1)^m \frac{\left(\frac{w}{2}\right)^{2m}}{(2m)!} \right] \\
U_{2n+1}(w,0) &= (-1)^n \left[\sin\frac{w}{2} - \sum_{m=0}^{n-1} (-1)^m \frac{\left(\frac{w}{2}\right)^{2m+1}}{(2m+1)!} \right] \\
U_{-n}(w,0) &= \cos\left(\frac{w}{2} + \frac{n\pi}{2}\right) \\
V_0(w,0) &= 1,\ V_{n+1}(w,0) = 0 \\
V_{-2n}(w,0) &= (-1)^n \sum_{m=0}^{n} (-1)^m \frac{\left(\frac{w}{2}\right)^{2m}}{(2m)!} \\
V_{-2n-1}(w,0) &= (-1)^n \sum_{m=0}^{n} (-1)^m \frac{\left(\frac{w}{2}\right)^{2m+1}}{(2m+1)!}.
\end{aligned}
$$

The computation of expressions (5) for not too small values of w may suffer from loss of digits owing to the alternating character of the series. This same objection arises in computing the alternating series (4) when z is not small.

We now turn to Zernike's method. Here Lommel's functions of two variables are expanded in products of Bessel functions

$$
(6)\qquad
\begin{aligned}
U_1(w,z) + iU_2(w,z) &= we^{\frac{1}{2}iw} \int_0^1 J_0(zt) e^{-\frac{1}{2}iwt^2} t\, dt \\
&= we^{\frac{1}{2}iw} \sum_{n=0}^{\infty} i^n (2n+1) \sqrt{\frac{2\pi}{w}}\, J_{n+\frac{1}{2}}\left(\frac{w}{4}\right) \frac{J_{2n+1}(z)}{z}.
\end{aligned}
$$

The essential advantage of this expansion is the following. In (6) all terms of the infinite sum have an absolute value smaller than 1 for all real values of w and z, so, contrary to Dekanosidze's method, there is no danger of loss of digits. This is readily proved by applying the recurrence formula for Bessel functions (Cf. [3], formula 3.2(1), p. 45)

$$2(2n+1)\frac{J_{2n+1}(z)}{z} = J_{2n}(z) + J_{2n+2}(z)$$

hence

$$\left| (2n+1)\frac{J_{2n+1}(z)}{z} \right| \leqq \frac{1}{2} \left| J_{2n}(z) \right| + \frac{1}{2} \left| J_{2n+2}(z) \right| \leqq 1.$$

Similarly for $J_{n+\frac{1}{2}}(x)$ the following integral representation is valid:

$$J_{n+\frac{1}{2}}(x) = (-i)^n \sqrt{\frac{x}{2\pi}} \int_{-1}^{+1} e^{ixt} P_n(t)\, dt,$$

(Cf. [3], formula 3.32(2), p. 50) which may be estimated by

$$|J_{n+\frac{1}{2}}(x)| \leqq 2\sqrt{\frac{x}{2\pi}} = \sqrt{\frac{2x}{\pi}};$$

hence

$$\left|\sqrt{\frac{2\pi}{w}}\, J_{n+\frac{1}{2}}\left(\frac{w}{4}\right)\right| \leqq \sqrt{\frac{2\pi}{w}}\sqrt{\frac{w}{2\pi}} = 1.$$

Another advantage is that U_1 and U_2 are calculated simultaneously because each of them is found by adding alternate terms of one single expansion.

The Bessel functions of odd and semi-odd order which are required in equation (6) may be computed very suitably by means of the recurrence technique developed by Goldstein and Thaler [6]. When computing a table of Lommel's functions on an electronic computer, it is possible to store these sequences of Bessel functions, after which various values of w and z may be combined to give $U_1(w, z)$ and $U_2(w, z)$.

The method may still be used, even for large values of w and z, though in that case a rather large sequence of Bessel functions must be computed.

A comparison of the two methods has been made for the case $w = 20$, $z = 20$. When Dekanosidze's method was followed for both functions $U_1(w, z)$ and $U_2(w, z)$, a total of 31 terms of the series in equation (4) had to be taken into account, each term being computed to twelve digits in order to obtain an accuracy of four decimal places (hence a loss of eight digits). When the method described here was followed, 11 terms were already sufficient to give simultaneously $U_1(w, z)$ and $U_2(w, z)$ with the same accuracy without any loss of digits.

Finally, the method described here has been used to recompute Lommel's original tables [4] (the functions $V_0(w, z)$ and $V_1(w, z)$ have been computed by equation (1)), the results being given below. In these tables, the decimals which deviate from Lommel's values have been italicized. Besides that, all values of $\frac{2}{w} V_1(w, z)$ differ by a factor -1 from Lommel's values because the definition of $V_n(w, z)$, as used in the present article and in [3], differs by a factor $(-1)^n$ from Lommel's original definition. (See the footnote at the bottom of [3], p. 537.)

Mathematical Institute
University of Groningen
The Netherlands

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TABLE OF LOMMEL'S FUNCTIONS OF TWO VARIABLES

$w = \pi$

z	$\frac{2}{w} U_1(w, z)$	$\frac{2}{w} U_2(w, z)$	$\frac{2}{w} V_0(w, z)$	$\frac{2}{w} V_1(w, z)$
0	+0.636620	+0.636620	+0.636620	0
1	+0.539802	+0.580638	+0.479744	−0.088772
2	+0.298890	+0.433460	+0.055002	−0.213022
3	+0.032426	+0.247396	−0.383136	−0.055402
4	−0.142282	+0.081868	−0.275023	+0.384893
5	−0.173625	−0.022258	+0.450640	+0.252583
6	−0.094496	−0.056474	+0.278235	−0.636026
7	+0.011819	−0.041090	−0.676734	−0.023425
8	+0.070711	−0.008787	+0.430319	+0.531656
9	+0.059110	+0.013939	−0.189447	−0.544147
10	+0.007050	+0.017334	+0.148505	+0.630010
11	−0.035803	+0.007209	−0.245499	−0.620117
12	−0.039518	−0.004302	+0.504943	+0.342522

$w = 2\pi$

z	$\frac{2}{w} U_1(w, z)$	$\frac{2}{w} U_2(w, z)$	$\frac{2}{w} V_0(w, z)$	$\frac{2}{w} V_1(w, z)$
0	0	+0.636620	+0.318310	0
1	−0.047572	+0.559947	+0.242644	−0.022268
2	−0.156737	+0.362318	+0.059998	−0.057118
3	−0.250135	+0.124194	−0.115909	−0.041158
4	−0.259807	−0.066375	−0.159699	+0.044515
5	−0.172632	−0.155073	−0.025675	+0.118189
6	−0.036806	−0.141277	+0.164916	+0.050182
7	+0.073194	−0.068276	+0.162949	−0.145567
8	+0.106459	+0.007067	−0.111169	−0.189077
9	+0.065125	+0.045577	−0.268535	+0.116651
10	−0.004675	+0.040714	+0.073684	+0.311923
11	−0.049843	+0.012197	+0.323900	−0.114359
12	−0.046338	−0.013333	−0.155667	−0.331053

$w = 3\pi$

z	$\frac{2}{w} U_1(w, z)$	$\frac{2}{w} U_2(w, z)$	$\frac{2}{w} V_0(w, z)$	$\frac{2}{w} V_1(w, z)$
0	−0.212207	+0.212207	+0.212207	0
1	−0.221811	+0.150853	+0.162106	−0.009903
2	−0.233157	−0.000541	+0.044154	−0.025710
3	−0.209291	−0.162866	−0.065351	−0.020817
4	−0.127691	−0.255583	−0.096321	+0.012549
5	−0.005131	−0.240383	−0.034488	+0.046240
6	+0.107300	−0.137968	+0.062157	+0.036719
7	+0.156654	−0.010133	+0.099345	−0.025132
8	+0.124279	+0.079105	+0.025841	−0.081135
9	+0.038751	+0.098281	−0.095896	−0.046848
10	−0.043777	+0.059355	−0.116648	+0.074775
11	−0.077056	+0.001898	+0.030683	+0.133189
12	−0.052825	−0.035554	+0.171787	−0.007646

Table of Lommel's Functions of Two Variables

$w = 4\pi$

z	$\frac{2}{w} U_1(w, z)$	$\frac{2}{w} U_2(w, z)$	$\frac{2}{w} V_0(w, z)$	$\frac{2}{w} V_1(w, z)$
0	0	0	+0.159155	0
1	+0.000759	−0.037360	+0.121669	−0.005572
2	+0.010698	−0.122929	+0.034215	−0.014526
3	+0.043564	−0.194789	−0.045730	−0.012219
4	+0.100101	−0.196607	−0.068629	+0.005486
5	+0.157469	−0.114657	−0.027959	+0.024001
6	+0.179329	+0.013349	+0.035306	+0.021696
7	+0.141278	+0.122492	+0.063628	−0.006592
8	+0.052246	+0.160931	+0.029137	−0.036976
9	−0.046242	+0.119653	−0.038977	−0.033318
10	−0.104762	+0.033666	−0.072886	+0.013463
11	−0.097782	−0.043578	−0.027365	+0.060545
12	−0.039653	−0.073719	+0.061663	+0.044025

$w = 5\pi$

z	$\frac{2}{w} U_1(w, z)$	$\frac{2}{w} U_2(w, z)$	$\frac{2}{w} V_0(w, z)$	$\frac{2}{w} V_1(u, z)$
0	+0.127324	+0.127324	+0.127324	0
1	+0.123693	+0.101421	+0.097369	−0.003566
2	+0.116978	+0.043947	+0.027779	−0.009316
3	+0.114163	+0.000633	−0.035346	−0.007972
4	+0.114193	+0.008654	−0.053424	+0.003028
5	+0.103761	+0.068242	−0.022719	+0.014668
6	+0.066399	+0.140571	+0.024568	+0.013915
7	−0.000892	+0.173623	+0.046307	−0.002302
8	−0.077848	+0.137780	+0.024054	−0.020595
9	−0.128770	+0.046262	−0.021724	−0.021117
10	−0.125074	−0.053370	−0.048087	+0.002141
11	−0.066712	−0.110067	−0.027077	+0.029850
12	+0.014393	−0.100916	+0.025355	+0.030738

$w = 6\pi$

z	$\frac{2}{w} U_1(w, z)$	$\frac{2}{w} U_2(w, z)$	$\frac{2}{w} V_0(w, z)$	$\frac{2}{w} V_1(u, z)$
0	0	+0.212207	+0.106103	0
1	−0.005291	+0.187222	+0.081156	−0.002477
2	−0.017713	+0.128841	+0.023335	−0.006476
3	−0.030684	+0.074204	−0.028890	−0.005594
4	−0.041775	+0.052871	−0.043819	+0.001917
5	−0.055411	+0.064625	−0.018991	+0.009906
6	−0.076993	+0.080250	+0.018958	+0.009616
7	−0.103193	+0.064885	+0.036479	−0.000963
8	−0.118371	+0.006400	+0.019824	−0.013120
9	−0.103083	−0.072681	−0.014730	−0.014204
10	−0.050141	−0.129002	−0.035334	−0.000298
11	+0.024503	−0.128183	−0.022325	+0.017291
12	+0.086795	−0.069153	+0.013475	+0.020232

Table of Lommel's Functions of Two Variables

$w = 7\pi$

z	$\frac{2}{w} U_1(w, z)$	$\frac{2}{w} U_2(w, z)$	$\frac{2}{w} V_0(w, z)$	$\frac{2}{w} V_1(w, z)$
0	−0.090946	+0.090946	+0.090946	0
1	−0.092742	+0.067502	+0.069570	−0.001820
2	−0.095331	+0.011836	+0.020097	−0.004761
3	−0.093184	−0.042950	−0.024469	−0.004136
4	−0.083669	−0.069547	−0.037187	+0.001326
5	−0.069496	−0.065233	−0.016277	+0.007149
6	−0.055115	−0.050886	+0.015516	+0.007029
7	−0.040559	−0.051440	+0.030185	−0.000452
8	−0.019618	−0.073600	+0.016738	−0.009122
9	+0.014183	−0.098796	−0.011166	−0.010150
10	+0.057961	−0.097344	−0.027951	−0.000826
11	+0.095375	−0.053091	−0.018474	+0.011275
12	+0.104159	+0.020684	+0.008673	+0.014010

$w = 8\pi$

z	$\frac{2}{w} U_1(w, z)$	$\frac{2}{w} U_2(w, z)$	$\frac{2}{w} V_0(w, z)$	$\frac{2}{w} V_1(w, z)$
0	0	0	+0.079578	0
1	+0.000190	−0.018684	+0.060878	−0.001393
2	+0.002679	−0.061687	+0.017639	−0.003647
3	+0.010993	−0.099549	−0.021243	−0.003179
4	+0.025878	−0.107904	−0.032324	+0.000974
5	+0.043375	−0.084168	−0.014231	+0.005408
6	+0.057603	−0.046848	+0.013178	+0.005359
7	+0.065631	−0.018864	+0.025804	−0.000228
8	+0.069350	−0.008873	+0.014458	−0.006731
9	+0.071904	−0.005808	−0.009042	−0.007608
10	+0.071829	+0.009152	−0.023198	−0.000876
11	+0.061297	+0.043412	−0.015655	+0.007971
12	+0.031978	+0.082866	+0.006317	+0.010231

	$w = 9\pi$		$w = 10\pi$	
z	$\frac{2}{w} U_1(w, z)$	$\frac{2}{w} U_2(w, z)$	$\frac{2}{w} U_1(w, z)$	$\frac{2}{w} U_2(w, z)$
0	+0.070736	+0.070736	0	+0.127324
1	+0.069624	+0.055367	−0.001905	+0.112361
2	+0.067676	+0.020712	−0.006385	+0.077695
3	+0.067323	−0.007570	−0.011132	+0.046173
4	+0.068669	−0.008852	−0.015445	+0.035955
5	+0.068172	+0.017625	−0.021256	+0.047323
6	+0.061099	+0.053530	−0.031103	+0.063679
7	+0.045681	+0.076475	−0.044829	+0.065343
8	+0.024884	+0.076750	−0.058321	+0.044755
9	+0.003842	+0.062426	−0.065889	+0.011066
10	−0.014690	+0.049474	−0.064357	−0.018761
11	−0.032149	+0.046026	−0.055052	−0.034099
12	−0.050776	+0.044629	−0.041669	−0.037907