it, or including that which we, $2c$ cycles before, deleted from it, until the (2^{k-1})st combination, which corresponds to the empty set plus element k.

Proof of (1). Since the binary representation of 2^{k-1} is a 1 bit followed by $(k-1)$ zeros, the kth element is included on cycle 2^{k-1}. The kth element will remain until the binary number 11 followed by $(k-1)$ zeros appears. This will be on cycle number $(2^k + 2^{k-1}) > (2^k - 1)$. Thus, all combinations from 2^{k-1} through $(2^k - 1)$ will include the kth element.

Proof of (2). Since $(2^{k-1} + c) + (2^{k-1} - c) = 2^k$, the binary representations of $(2^{k-1} + c)$ and $(2^{k-1} - c)$ correspond in all their low-order zeros, and the low-order 1, in which they also correspond. The bit above the 1 must differ in the two numbers, due to the binary carry. Thus, $B(2^{k-1} + c) = -B(2^{k-1} - c)$.

To complete the proof by induction, we may note, by Table 1, that the algorithm has generated all combinations for $k \leqq 4$.

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Generation of Permutations by Addition

By John R. Howell

1. Introduction. Suppose one wishes to generate the $k!$ permutations of k distinct marks. Representing these k marks by $0, 1, 2, \cdots, (k-1)$ written side by side to form the "digits" of a base k integer, then the repeated addition of 1 will generate integers whose "digits" represent permutations of k marks. Many numbers are also generated which are not permutations. D. H. Lehmer [2] states that this so-called addition method can be made more efficient by adding more than 1 to each successive integer.

2. Method. In this note, we show that the correct number greater than 1 to add to this integer is a multiple of $(k-1)$ radix k.

LEMMA 1. *The arithmetic difference radix k between an integer composed of mutually unlike digits and another integer composed of a permutation of the same digits is a multiple of $(k-1)$.*

Considering the process of "casting out nines," it is obvious that the two integers are congruent mod $(k-1)$. Hence, their difference is zero mod $(k-1)$.

The method seems to have two advantages. First, one can generate all $k!$ permutations in lexicographic order. Second, all permutations "between" two given permutations can be obtained. The process can be made to be cyclic if upon obtaining $(k-1), \cdots, 0$ one takes the next permutation to be $0, 1, \cdots, (k-1)$.

3. Example. Suppose we wish to generate the 4! permutations of 4 marks. Representing these 4 marks by 0, 1, 2 and 3, we add 3 radix 4 to 0123 to get 0132. Continuing this process we get the 4! permutations desired. The array below shows

the first 16 numbers generated by this process. An asterisk marks each integer whose digits represent a required permutation. The other integers were rejected because of the occurrence of repeated digits.

Sequence	Integer	Sequence	Integer
1	0123*	9	0303
2	0132*	10	0312*
3	0201	11	0321*
4	0210	12	0330
5	0213*	13	0333
6	0222	14	1002
7	0231*	15	1011
8	0300	16	1020

4. Adaptation to a Computer. In a computer such as the IBM 7090 where convert instructions are available it is easy to do radix k arithmetic. Otherwise one could simulate the process by adding 9 digit-wise and testing the resulting sum for having unique digits each one of which is one of the original k digits.

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Multiple Quadrature with Central Differences on One Line

By Herbert E. Salzer

Abstract. The coefficients A_{2m}^{n} in the n-fold quadrature formulas for the stepwise integration of (1) $y^{(n)} = f(x, y, y', \cdots, y^{(n-1)})$, at intervals of h, namely, for n even, (2) $\delta^n y_0 = h^n \sum_{m=1}^{10} (1 + A_{2m}^n \delta^{2m}) f_0 + \cdots$, for n odd, (3) $\mu \delta^n y_0 = h^n \sum_{m=1}^{10}$ $(1 + A_{2m}^n \delta^{2m}) f_0 + \cdots$, are tabulated exactly for $n = 1(1)6$, $m = 1(1)10$. They were calculated from the well-known symbolic formulas (4) $\delta^n y = (\delta/D)^n f$, (5) $(\delta/D)^n = (\delta h/2 \sinh^{-1}(\delta/2))^n$ and (6) $\mu = (1 + \delta^2/4)^{1/2} = 1 + \dfrac{\delta^2}{8} - \dfrac{\delta^4}{128} + \dfrac{\delta^6}{1024} -$