# Some Relations and Values For the Generalized Riemann Zeta Function

By Eldon R. Hansen and Merrell L. Patrick

**1. Introduction.** In this paper we derive several relations among values of the generalized Riemann Zeta function $\zeta(s, a)$. We show that there is a "fundamental domain" for the variables $s$ and $a$ so that values outside this domain can be obtained from those inside by algebraic operations, using values of the Gamma and trigonometric functions.

We give a table of values of the Zeta function which were calculated using these relations to supplement the formulas commonly used for such calculations. Table 1 gives $\zeta(s, a)$ to seventeen decimal places for $a = \frac{1}{4}$ and $\frac{3}{4}$ and for $s = -\frac{61}{3}(\frac{1}{3})\frac{64}{3}$. The ordinary Riemann Zeta function is also given to seventeen decimal places for $s = 0(\frac{1}{3})\frac{64}{3}$ in Table 2. These tables are useful in diffraction theory [10], [11].

**2. Relations among Zeta Functions.** Consider the defining relation

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s} \tag{1}$$

which holds for $\text{Re}(s) > 1$ and all values of $a$ except zero or negative integers. (One usually assumes $s$ complex but $a$ real.) We first note that

$$\begin{aligned} \zeta(s, a + 1) &= \sum_{n=0}^{\infty} \frac{1}{(n + a + 1)^s} \\ &= \sum_{n=0}^{\infty} \frac{1}{(n + a)^s} - \frac{1}{a^s} \\ &= \zeta(s, a) - a^{-s}. \end{aligned} \tag{2}$$

By analytic continuation, this relation holds for all $s$. Thus if $\zeta(s, a)$ is known for all $s$, but only for values of $a$ satisfying $a_0 < a \leqq a_0 + 1$ for some $a_0$, then we can calculate $\zeta(s, a)$ for any value of $a$ by algebraic operations. One usually specifies $a_0 = 0$, so that the "fundamental interval" is $0 < a \leqq 1$.

We now show that this fundamental interval in $a$ need only be of length $\frac{1}{2}$. In the defining relation (1), let $a = (1/q) - b$ where $q$ is an integer. Then

$$\begin{aligned} \zeta\left(s, \frac{1}{q} - b\right) &= q^s \sum_{n=0}^{\infty} \frac{1}{(nq + 1 - bq)^s} \\ &= q^s \left[\sum_{n=0}^{\infty} \frac{1}{(n + 1 - bq)^s} - \sum_{r=2}^{q} \sum_{n=0}^{\infty} \frac{1}{(nq + r - bq)^s}\right] \\ &= q^s \zeta(s, 1 - bq) - \sum_{r=2}^{q} \zeta\left(s, \frac{r}{q} - b\right). \end{aligned}$$

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Therefore

$$\sum_{r=1}^{q} \zeta\left(s, \frac{r}{q} - b\right) = q^s \zeta(s, 1 - bq). \tag{3}$$

By analytic continuation, this result holds for all $s$. This relation was obtained by Powell [7] for the special case $s = \frac{1}{2}$.

Equation (3) yields many interesting and useful results. For example, let $b = \frac{1}{2} - c$. Then, for $q = 2$, we obtain

$$\zeta(s, c) = 2^s \zeta(s, 2c) - \zeta(s, c + \tfrac{1}{2}). \tag{4}$$

For $c = \frac{1}{2}$ this yields the well known result

$$\zeta(s, \tfrac{1}{2}) = (2^s - 1)\zeta(s), \tag{5}$$

where $\zeta(s)$ is the ordinary Riemann Zeta function defined by $\zeta(s, 1) = \zeta(s)$.

Now assume that $\zeta(s, a)$ is known for all $s$ and for all $a$ in the interval $\frac{1}{2} < a \leqq 1$. Then equation (4) shows that, if $a_1$ is such that $0 < a_1 \leqq \frac{1}{2}$, then $\zeta(s, a_1)$ can be expressed in terms of values assumed known. We merely apply relation (4) repeatedly as needed. An inductive proof can be easily obtained but will not be given here. As an example of the use of equation (4), consider the case $a_1 = \frac{1}{8}$. From equation (4),

$$\zeta(s, \tfrac{1}{8}) = 2^s \zeta(s, \tfrac{1}{4}) - \zeta(s, \tfrac{5}{8}). \tag{6}$$

The argument $\frac{1}{4}$ is not in the desired interval, so we use (4) to obtain

$$\zeta(s, \tfrac{1}{4}) = 2^s \zeta(s, \tfrac{1}{2}) - \zeta(s, \tfrac{3}{4}). \tag{7}$$

Again, we do not know $\zeta(s, \frac{1}{2})$, so we use (4) to obtain

$$\zeta(s, \tfrac{1}{2}) = 2^s \zeta(s, 1) - \zeta(s, 1) = (2^s - 1)\zeta(s). \tag{8}$$

Combining (6), (7) and (8), we have

$$\zeta(s, \tfrac{1}{8}) = 2^{2s}(2^s - 1)\zeta(s) - 2^s \zeta(s, \tfrac{3}{4}) - \zeta(s, \tfrac{5}{8}).$$

All the functions in the right member are assumed known, hence $\zeta(s, \frac{1}{8})$ is obtained by algebraic operations.

For special values of $a$, the more general relation (3) may be used more fruitfully than the special form (4).

We now see that $\frac{1}{2} < a \leqq 1$ is a fundamental interval for $\zeta(s, a)$. We next show that for rational values of $a$, we can obtain $\zeta(s, a)$ for $\operatorname{Re}(s) < \frac{1}{2}$ from values for $\operatorname{Re}(s) \geqq \frac{1}{2}$.

Consider the Hurwitz formula (see [1], page 269)

$$\frac{(2\pi)^s}{2\Gamma(s)} \zeta(1 - s, a) = \cos \frac{\pi s}{2} \sum_{n=1}^{\infty} \frac{\cos 2\pi n a}{n^s} + \sin \frac{\pi s}{2} \sum_{n=1}^{\infty} \frac{\sin 2\pi n a}{n^s}, \tag{9}$$

which is valid for $\operatorname{Re}(s) > 1$ and $0 < a \leqq 1$. Let $a = p/q$, where $p$ and $q$ are integers (which need not be relatively prime). Then

$$\frac{(2\pi)^s}{2\Gamma(s)}\zeta\left(1-s,\frac{p}{q}\right) = \cos\frac{\pi s}{2}\sum_{n=1}^{\infty}\frac{\cos 2\pi np/q}{n^s} + \sin\frac{\pi s}{2}\sum_{n=1}^{\infty}\frac{\sin 2\pi np/q}{n^s}$$

$$= \cos\frac{\pi s}{2}\sum_{r=0}^{q-1}\sum_{n=1}^{\infty}\frac{\cos 2\pi(qn-r)p/q}{(qn-r)^s} + \sin\frac{\pi s}{2}\sum_{r=0}^{q-1}\sum_{n=1}^{\infty}\frac{\sin 2\pi(qn-r)p/q}{(qn-r)^s}$$

$$= \frac{1}{q^s}\cos\frac{\pi s}{2}\sum_{r=0}^{q-1}\sum_{n=1}^{\infty}\frac{\cos 2\pi rp/q}{(n-r/q)^s} - \frac{1}{q^s}\sin\frac{\pi s}{2}\sum_{r=0}^{q-1}\sum_{n=1}^{\infty}\frac{\sin 2\pi rp/q}{(n-r/q)^s}$$

$$= \frac{1}{q^s}\sum_{r=0}^{q-1}\left[\cos\frac{\pi s}{2}\cos\frac{2\pi rp}{q} - \sin\frac{\pi s}{2}\sin\frac{2\pi rp}{q}\right]\sum_{n=0}^{\infty}\frac{1}{(n+1-r/q)^s}.$$

Replacing $r$ by $q - r$ and simplifying,

$$\zeta(1-s, p/q) = \frac{2\Gamma(s)}{(2\pi q)^s}\sum_{r=1}^{q}\cos\pi\left(\frac{s}{2}-\frac{2rp}{q}\right)\zeta(s, r/q). \tag{10}$$

By analytic continuation, this result holds for all $s$. Thus, if $\zeta(s, a)$ is known for $\mathrm{Re}(s) \geqq \frac{1}{2}$ and all rational $a$, we can compute $\zeta(s, a)$ for all $s$ and all rational $a$, $0 < a \leqq 1$, from equation (10). Conversely, if we know $\zeta(s, a)$ for all rational $a$, $0 < a \leqq 1$, and for $\mathrm{Re}(s) \leqq \frac{1}{2}$, we can compute $\zeta(s, a)$ from equation (10) for; all $s$ except for integer values. The right hand side of equation (10) contains $\Gamma(s)$, which is infinite for $s$ a negative integer. Hence equation (10) cannot, in general, be used directly to yield $\zeta(s, a)$ for $s$ a positive integer. However equations (2), (4), and (10) enable us to find $\zeta(s, a)$ for all $s$ and all rational $a$ from values of $\zeta(s, a)$ for $\mathrm{Re}(s) \geqq \frac{1}{2}$ and $\frac{1}{2} < a \leqq 1$.

Let us now consider some special cases of equation (10). For $p = 1$ and $q = 2$, we again obtain equation (5). For $p = q = 1$, we obtain the Riemann relation

$$\zeta(1-s) = \frac{2\Gamma(s)}{(2\pi)^s}\cos\frac{\pi s}{2}\zeta(s). \tag{11}$$

For the general case $p = q$, we obtain, by using relation (11),

$$\sum_{r=1}^{q-1}\zeta(s, r/q) = (q^s - 1)\zeta(s) \tag{12}$$

which also follows from equation (3). If we also write equation (12) with $s$ replaced by $1 - s$, then the resulting equation can be combined with equations (11) and (12) to yield

$$\pi^{-s/2}\Gamma(s/2)(q^s-1)\sum_{r=1}^{q-1}\zeta(s, r/q) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)(q^{1-s}-1)\sum_{r=1}^{q-1}\zeta(1-s, r/q). \tag{13}$$

That is, the members of this equation are invariant under replacement of $s$ by $1 - s$. Hence we may regard equation (13) as a generalization of the Riemann relation (11), which is often written as

$$\Gamma\left(\frac{1-s}{2}\right)\pi^{-(1-s)/2}\zeta(1-s) = \Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s).$$

Next substitute $q = 4$ and $p = 1$ and 3 in equation (10). In this case, we obtain using equations (5) and (11)

$$(14)\quad \zeta(1-s, \tfrac{1}{2} \pm \tfrac{1}{4}) = \frac{2-2^s}{4^s}\zeta(1-s) \pm \frac{2\Gamma(s)}{(8\pi)^s}\sin\frac{\pi s}{2}[\zeta(s, \tfrac{3}{4}) - \zeta(s, \tfrac{1}{4})].$$

Subtracting this equation, using the lower sign from the equation using the uppe sign, we obtain

$$(15)\quad \zeta(1-s, \tfrac{3}{4}) - \zeta(1-s, \tfrac{1}{4}) = \frac{4\Gamma(s)}{(8\pi)^s}\sin\frac{\pi s}{2}[\zeta(s, \tfrac{3}{4}) - \zeta(s, \tfrac{1}{4})].$$

Now the Dirichlet $L$-function is defined to be

$$L(s) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^s}$$

for $\mathrm{Re}(s) > 1$. This can be expressed as

$$L(s) = 4^{-s}[\zeta(s, \tfrac{1}{4}) - \zeta(s, \tfrac{3}{4})].$$

Hence equation (15) can be rewritten as

$$L(1-s) = (2/\pi)^s\Gamma(s)\sin\frac{\pi s}{2}L(s).$$

This relation is derived (but misprinted) by Titchmarsh (see [6], page 66).

For $m$ equal to a positive integer,

$$(16)\quad \zeta(-m, a) = -\frac{B_{m+1}(a)}{(m+1)},$$

where $B_m(a)$ is the $m$th Bernoulli polynomial in $a$. (See [1], page 267.) Henc letting $s \to -2m$ in equation (14), and using the fact that $B_m(1-a$ $= (-1)^m B_m(a)$, we obtain

$$\zeta(2m+1, \tfrac{1}{2} \pm \tfrac{1}{4}) = 2^{2m}(2^{2m+1}-1)\zeta(2m+1) \pm \frac{2^{6m+1}\pi^{2m+1}}{(2m+1)!}B_{2m+1}(\tfrac{1}{4}).$$

From these equations, we find that

$$L(2m+1) = -\frac{(2\pi)^{2m+1}}{2(2m+1)!}B_{2m+1}(\tfrac{1}{4}).$$

Many other interesting results can be obtained by looking at special cases equation (10).

The method used to derive equations (3) and (10) can also be applied to tl more general function

$$\zeta(s, a, z) = \sum_{n=0}^{\infty}\frac{z^n}{(n+a)^s}$$

discussed by Mitchell [8], Lerch [9] and others.

**3. Calculation of the Table.** In the discussion above we showed, by using equation (10), that we could find $\zeta(s, a)$ for the $\mathrm{Re}(s) \leqq \frac{1}{2}$ and rational $a$, $0 < a \leqq 1$, provided we know $\zeta(s, a)$ for $\mathrm{Re}(s) > \frac{1}{2}$ and rational $a$, $0 < a \leqq 1$. The equation

$$\zeta(s, a) = \sum_{n=0}^{N-1} \frac{1}{(n + a)^s} - \frac{(N + a)^{1-s}}{1 - s} + \frac{(N + a)^{-s}}{2} - \sum_{n=1}^{\infty} \frac{B_n}{(2n)!} \frac{\Gamma(1 - s)}{\Gamma(2 - s - 2n)} (N + a)^{-s-(2n-1)}, \tag{17}$$

where $B_1 = \frac{1}{6}$, $B_2 = -\frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = -\frac{1}{30}$, $B_5 = \frac{5}{66}$, $\cdots$ are Bernoulli numbers, is a representation of $\zeta(s, a)$ for $\mathrm{Re}(s) > 0$, $s \neq 1$, and $0 < a \leqq 1$. We use this asymptotic series to calculate $\zeta(s, \frac{1}{4})$, $\zeta(s, \frac{3}{4})$, and $\zeta(s, 1) = \zeta(s)$ for $s = \frac{2}{3}(\frac{1}{3})\frac{64}{3}$. It is not necessary to use this expression to calculate $\zeta(s, \frac{1}{2})$ because of equation (8). With these values of $\zeta(s, \frac{1}{4})$, $\zeta(s, \frac{3}{4})$, and $\zeta(s, 1)$ for $s = \frac{2}{3}(\frac{1}{3})\frac{64}{3}$, we calculate $\zeta(s, \frac{1}{4})$ and $\zeta(s, \frac{3}{4})$, using (14), for $s = -\frac{61}{3}(\frac{1}{3})\frac{1}{3}$ with the exception $s = 0$. For $s = 0$ we use the identity $\zeta(0, a) = \frac{1}{2} - a$. The Gamma function, $\Gamma(s)$, appearing on the right side of equation (10) is calculated by making use of the asymptotic series (see [1] Section 12.33)

$$\log \Gamma(s) = (s - \tfrac{1}{2}) \log s - s + \tfrac{1}{2} \log (2\pi) + \sum_{r=1}^{\infty} \frac{(-1)^{r-1} B_r}{2r(2r - 1)s^{2r-1}}$$

where $B_r$ are the Bernoulli numbers. This expression holds for $|\arg s| \leqq \frac{1}{2}\pi - \Delta$ and $0 < \Delta < \frac{1}{4}\pi$. Using this relation in conjunction with the equation $\Gamma(s + 1) = s\Gamma(s)$, we obtain the desired values of the Gamma function.

Equation (17), for $a = 1$, is equivalent to the expression used by Gram [2] and by Haselgrove [3] in preparing their tables of the ordinary Riemann Zeta function.

**4. Methods of Checking the Table Values.** Equation (16) was used to check the table entries for negative integer values of $s$, while Hurwitz's formula (9) was used to check the entries for $s = -\frac{61}{3}(\frac{1}{3}) - \frac{28}{3}$. For positive integer values of $s$, the values of $\zeta(s)$ agree exactly with the table in [5], page xxv. Our table was not checked for other values of $s$, since no other simple method seemed to present itself. Note, however, the table for negative values of $s$ was calculated using the entries for positive values of $s$. Hence, it seems safe to assume that the table entries for positive values of $s$ are correct to at least as many digits as the table entries for corresponding negative values of $s$.

**5. Comments Regarding the Calculation of the Table.** The table shown below was calculated using a precision of seventy binary digits, which is approximately equivalent to twenty-one decimal digits.

It was impossible to enter most of the values of $s$ exactly in the computer, since 1/3 does not have a finite binary representation. Therefore, we should consider the errors introduced in our table from using a truncated binary representation. We were able to show, by using a truncated Taylor series, that the errors so introduced did not affect the accuracy of the table. The various calculations involved in showing the insignificance of these errors were not included in this paper, since they were quite straightforward.

**6. The Table of Values.** The tabular data have been listed to seventeen significant decimal digits. The decimal point is located immediately to the left of the left-most digit of each entry, while the two-digit integer at the right denotes the exponent of the power of ten by which the entry is to be multiplied.

Lockheed Missiles and Space Company
Palo Alto, California

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TABLE 1

| $s$ | $\zeta(s, \frac{1}{4})$ | | $\zeta(s, \frac{3}{4})$ | |
|---|---|---|---|---|
| 64/3 | 69814 63658 33156 77 | 13 | 46276 41642 09957 54 | 03 |
| 21 | 43980 46511 10400 92 | 13 | 42044 91481 42985 79 | 03 |
| 62/3 | 27705 95688 87832 77 | 13 | 38200 34049 99724 72 | 03 |
| 61/3 | 17453 65914 58290 05 | 13 | 34707 31296 92456 16 | 03 |
| 20 | 10995 11627 77601 15 | 13 | 31533 68689 82531 57 | 03 |
| 59/3 | 69264 89222 19591 87 | 12 | 28650 25632 40314 70 | 03 |
| 58/3 | 43634 14786 45735 82 | 12 | 26030 48587 06422 96 | 03 |
| 19 | 27487 79069 44014 41 | 12 | 23650 26655 19854 06 | 03 |
| 56/3 | 17316 22305 54910 39 | 12 | 21487 69390 40321 69 | 03 |
| 55/3 | 10908 53696 61447 33 | 12 | 19522 86640 47054 77 | 03 |
| 18 | 68719 47673 60180 15 | 11 | 17737 70232 63351 39 | 03 |
| 53/3 | 43290 55763 87431 22 | 11 | 16115 77333 52502 14 | 03 |
| 52/3 | 27271 34241 53785 57 | 11 | 14642 15330 71921 73 | 03 |
| 17 | 17179 86918 40225 19 | 11 | 13303 28096 72635 09 | 03 |
| 50/3 | 10822 63940 97051 87 | 11 | 12086 83509 03545 36 | 03 |
| 49/3 | 68178 35603 86554 48 | 10 | 10981 62111 35873 48 | 03 |
| 16 | 42949 67296 02814 98 | 10 | 99774 68117 34460 20 | 02 |
| 47/3 | 27056 59852 45055 63 | 10 | 90651 35226 87845 72 | 02 |
| 46/3 | 17044 58900 99251 96 | 10 | 82362 26573 20047 47 | 02 |
| 15 | 10737 41824 03518 96 | 10 | 74831 14030 73469 44 | 02 |
| 44/3 | 67641 49631 42966 14 | 09 | 67988 67020 79914 81 | 02 |
| 43/3 | 42611 47252 80800 17 | 09 | 61771 88735 02511 35 | 02 |
| 14 | 26843 54560 43992 27 | 09 | 56123 58191 92793 72 | 02 |
| 41/3 | 16910 37408 23656 51 | 09 | 50991 77593 55328 03 | 02 |
| 40/3 | 10652 86813 61045 94 | 09 | 46329 24497 93384 86 | 02 |
| 13 | 67108 86405 50022 12 | 08 | 42093 08367 45511 12 | 02 |
| 38/3 | 42275 93525 33217 14 | 08 | 38244 31093 55480 72 | 02 |
| 37/3 | 26632 17039 13370 57 | 08 | 34747 51134 84446 15 | 02 |
| 12 | 16777 21606 87796 31 | 08 | 31570 50939 08321 59 | 02 |
| 35/3 | 10568 98387 26213 03 | 08 | 28684 07349 76745 21 | 02 |
| 34/3 | 66580 42661 71997 90 | 07 | 26061 64725 71060 80 | 02 |
| 11 | 41943 04086 03548 00 | 07 | 23679 10527 14066 46 | 02 |
| 32/3 | 26422 46042 34022 87 | 07 | 21514 53144 78372 37 | 02 |
| 31/3 | 16645 10745 38096 69 | 07 | 19548 01769 36192 33 | 02 |
| 10 | 10485 76107 68311 48 | 07 | 17761 48118 13370 04 | 02 |
| 29/3 | 66056 16034 78926 59 | 06 | 16138 49852 76111 40 | 02 |
| 28/3 | 41612 77864 98520 10 | 06 | 14664 15539 12355 32 | 02 |
| 9 | 26214 41349 21724 08 | 06 | 13324 91014 94593 48 | 02 |
| 26/3 | 16514 05173 61085 33 | 06 | 12108 47045 54080 94 | 02 |
| 25/3 | 10403 20723 10878 44 | 06 | 11003 68161 90407 28 | 02 |
| 8 | 65536 16938 67090 17 | 05 | 10000 42589 27890 58 | 02 |
| 23/3 | 41285 27582 34437 32 | 05 | 90895 31887 04304 92 | 01 |
| 22/3 | 26008 17634 03920 72 | 05 | 82626 93498 64416 39 | 01 |
| 7 | 16384 21345 51995 72 | 05 | 75123 97920 96579 80 | 01 |
| 20/3 | 10321 50411 38500 09 | 05 | 68318 62538 56294 03 | 01 |
| 19/3 | 65022 44664 70976 70 | 04 | 62149 80828 62956 03 | 01 |
| 6 | 40962 70948 06401 08 | 04 | 56562 77857 28816 43 | 01 |
| 17/3 | 25806 12469 70992 37 | 04 | 51508 76680 73079 38 | 01 |
| 16/3 | 16258 18668 37913 66 | 04 | 46944 78141 49920 25 | 01 |
| 5 | 10243 48974 52658 06 | 04 | 42833 58575 64239 47 | 01 |
| 14/3 | 64546 14107 47051 67 | 03 | 39143 93590 59400 43 | 01 |
| 13/3 | 40679 42509 60218 28 | 03 | 35851 22960 35115 28 | 01 |

TABLE 1—*Continued*

| $s$ | $\zeta(s, \frac{1}{4})$ | | $\zeta(s, \frac{3}{4})$ | |
|---|---|---|---|---|
| 4 | 25646 36906 68198 07 | 03 | 32938 85422 47510 00 | 01 |
| 11/3 | 16178 62720 13961 71 | 03 | 30400 81320 64265 08 | 01 |
| 10/3 | 10217 48139 83634 30 | 03 | 28246 87653 98801 74 | 01 |
| 3 | 64663 86996 87684 60 | 02 | 26513 16608 16881 98 | 01 |
| 8/3 | 41092 64168 75447 99 | 02 | 25285 74853 60394 04 | 01 |
| 7/3 | 26334 77863 02380 64 | 02 | 24759 96055 02275 82 | 01 |
| 2 | 17197 32915 45071 11 | 02 | 25418 79647 67160 65 | 01 |
| 5/3 | 11787 30284 47967 72 | 02 | 28747 02273 10830 06 | 01 |
| 4/3 | 95676 33344 56813 25 | 01 | 42231 01605 70504 20 | 01 |
| 1 | $\infty$ | | $\infty$ | |
| 2/3 | −24411 86144 96889 78 | 00 | −20381 06131 82465 56 | 01 |
| 1/3 | 33101 39009 27282 69 | 00 | −64976 99170 75290 43 | 00 |
| 0 | 25000 00000 00000 00 | 00 | −25000 00000 00000 00 | 00 |
| −1/3 | 13499 19957 79665 53 | 00 | −89579 84483 88113 98 | −01 |
| −2/3 | 55399 96817 77458 36 | −01 | −19221 97815 54288 52 | −01 |
| −1 | 10416 66666 66666 67 | −01 | 10416 66666 66666 67 | −01 |
| −4/3 | −10558 77278 36124 00 | −01 | 20147 85800 02319 75 | −01 |
| −5/3 | −16992 51822 29812 52 | −01 | 20093 86425 54835 96 | −01 |
| −2 | −15625 00000 00000 00 | −01 | 15625 00000 00000 00 | −01 |
| −7/3 | −10834 45058 95508 42 | −01 | 97268 16010 89067 10 | −02 |
| −8/3 | −52815 71878 70004 44 | −02 | 40704 00253 32719 07 | −02 |
| −3 | −45572 91666 66666 67 | −03 | −45572 91666 66666 67 | −03 |
| −10/3 | 29301 12394 80509 84 | −02 | −34597 28641 05474 40 | −02 |
| −11/3 | 46698 19500 95810 35 | −02 | −48805 58903 88315 47 | −02 |
| −4 | 48828 12500 00000 00 | −02 | −48828 12500 00000 00 | −02 |
| −13/3 | 38899 36684 63823 16 | −02 | −37822 44498 22381 09 | −02 |
| −14/3 | 21239 95040 97140 41 | −02 | −19861 15110 11172 00 | −02 |
| −5 | 60066 34424 60317 46 | −04 | 60066 34424 60317 46 | −04 |
| −16/3 | −18432 75029 83014 06 | −02 | 19233 18930 79122 45 | −02 |
| −17/3 | −31960 86810 98371 30 | −02 | 32323 03687 52690 05 | −02 |
| −6 | −37231 44531 25000 00 | −02 | 37231 44531 24500 00 | −02 |
| −19/3 | −33017 11705 70240 87 | −02 | 32782 87245 91040 61 | −02 |
| −20/3 | −19865 27145 97911 51 | −02 | 19530 95592 98213 26 | −02 |
| −7 | −16148 88509 11458 33 | −04 | −16148 88509 11458 33 | −04 |
| −22/3 | 22055 37817 57697 42 | −02 | −22292 83934 16781 00 | −02 |
| −23/3 | 41577 48345 17227 95 | −02 | −41695 51768 48320 96 | −02 |
| −8 | 52833 55712 89062 50 | −02 | −52833 55712 89062 50 | −02 |
| −25/3 | 50987 50282 80912 53 | −02 | −50896 45120 98702 91 | −02 |
| −26/3 | 33229 98874 70615 11 | −02 | −33088 82387 94668 38 | −02 |
| −9 | 73837 51146 72111 74 | −05 | 73837 51146 72111 74 | −05 |
| −28/3 | −43631 40309 49183 10 | −02 | 43748 62335 98758 29 | −02 |
| −29/3 | −88331 79840 49455 97 | −02 | 88394 53105 86276 23 | −02 |
| −10 | −12045 14503 47900 39 | −01 | 12045 14503 47900 39 | −01 |
| −31/3 | −12448 12326 45933 15 | −01 | 12442 55605 16962 73 | −01 |
| −32/3 | −86654 89190 70478 86 | −02 | 86562 63379 78600 91 | −02 |
| −11 | −51470 93963 85593 44 | −05 | −51470 93963 85593 44 | −05 |
| −34/3 | −12956 24917 98495 48 | −01 | −12964 94737 89903 23 | −01 |
| −35/3 | 27851 69352 45565 33 | −01 | −27856 63944 24212 63 | −01 |
| −12 | 40274 33693 40896 61 | −01 | −40274 33693 40896 61 | −01 |
| −37/3 | 44066 74225 89134 15 | −01 | −44061 81300 66818 43 | −01 |
| −38/3 | 32424 26287 38385 85 | −01 | −32415 62726 12390 43 | −01 |
| −13 | 50856 42139 11692 30 | −05 | 50856 42139 11692 30 | −05 |
| −40/3 | −53981 95160 35609 58 | −01 | 53991 01083 71719 25 | −01 |

TABLE 1—*Continued*

| $s$ | $\zeta(s, \frac{1}{4})$ | | $\zeta(s, \frac{3}{4})$ | |
|---|---|---|---|---|
| −41/3 | −12213 66720 37044 43 | 00 | 12214 20945 65951 94 | 00 |
| −14 | −18566 93821 02787 49 | 00 | 18566 93821 02787 49 | 00 |
| −43/3 | −21331 38708 22892 45 | 00 | 21330 79048 86111 17 | 00 |
| −44/3 | −16461 18663 02940 22 | 00 | 16460 09042 37713 87 | 00 |
| −15 | −67634 01438 99342 01 | −05 | −67634 01438 99342 01 | −05 |
| −46/3 | 30053 84266 68572 82 | 00 | −30055 10353 95330 80 | 00 |
| −47/3 | 71088 87471 07870 00 | 00 | −71089 66375 73979 33 | 00 |
| −16 | 11287 34563 53554 50 | 01 | −11287 34563 53554 50 | 01 |
| −49/3 | 13532 13201 24995 85 | 01 | −13532 03739 35347 35 | 01 |
| −50/3 | 10887 19301 23970 26 | 01 | −10887 01175 14453 29 | 01 |
| −17 | 11649 82235 12560 03 | −04 | 11649 82235 12560 03 | −04 |
| −52/3 | −21552 81617 38443 47 | 01 | 21553 04222 79286 09 | 01 |
| −53/3 | −53020 89805 52583 77 | 01 | 53021 04518 12564 35 | 01 |
| −18 | −87489 01292 85995 50 | 01 | 87489 01292 85995 50 | 01 |
| −55/3 | −10892 53279 53442 03 | 02 | 10892 51375 45982 56 | 02 |
| −56/3 | −90944 16979 46386 51 | 01 | 90943 79125 83854 77 | 01 |
| −19 | −25230 56188 58262 09 | −04 | −25230 56188 58262 09 | −04 |
| −58/3 | 19350 58818 71838 18 | 02 | −19350 63892 61704 53 | 02 |
| −59/3 | 49304 35983 54058 91 | 02 | −49304 39403 87055 58 | 02 |
| −20 | 84212 65834 78432 17 | 02 | −84212 65834 78432 17 | 02 |
| −61/3 | 10846 36397 60161 63 | 03 | −10846 35923 59976 00 | 03 |

TABLE 2

| s | ζ(s) | s | ζ(s) |
|---|---|---|---|
| 64/3 | 10000 00378 53233 25 01 | 31/3 | 10007 87512 59735 22 01 |
| 21 | 10000 00476 93298 68 01 | 10 | 10009 94575 12781 81 01 |
| 62/3 | 10000 00600 91541 53 01 | 29/3 | 10012 56544 89496 33 01 |
| 61/3 | 10000 00757 13123 99 01 | 28/3 | 10015 88198 62306 52 01 |
| 20 | 10000 00953 96203 39 01 | 9 | 10020 08392 82608 22 01 |
| 59/3 | 10000 01201 96944 13 01 | 26/3 | 10025 41240 15797 39 01 |
| 58/3 | 10000 01514 46249 87 01 | 25/3 | 10032 17649 68406 42 01 |
| 19 | 10000 01908 21271 66 01 | 8 | 10040 77356 19794 43 01 |
| 56/3 | 10000 02404 35545 91 01 | 23/3 | 10051 71613 23892 26 01 |
| 55/3 | 10000 03029 52623 90 01 | 22/3 | 10065 66797 08110 32 01 |
| 18 | 10000 03817 29326 50 01 | 7 | 10083 49277 38192 28 01 |
| 53/3 | 10000 04809 96364 94 01 | 20/3 | 10106 32075 32048 38 01 |
| 52/3 | 10000 06060 86098 24 01 | 19/3 | 10135 64087 21870 02 01 |
| 17 | 10000 07637 19763 79 01 | 6 | 10173 43061 98444 91 01 |
| 50/3 | 10000 09623 69761 94 01 | 17/3 | 10222 34193 45949 71 01 |
| 49/3 | 10000 12127 16680 42 01 | 16/3 | 10285 97325 52868 25 01 |
| 16 | 10000 15282 25940 87 01 | 5 | 10369 27755 14336 99 01 |
| 47/3 | 10000 19258 75533 22 01 | 14/3 | 10479 19229 28351 53 01 |
| 46/3 | 10000 24270 74650 59 01 | 13/3 | 10625 64589 86109 19 01 |
| 15 | 10000 30588 23630 70 01 | 4 | 10823 23233 71113 82 01 |
| 44/3 | 10000 38551 79067 62 01 | 11/3 | 11094 13692 59748 37 01 |
| 43/3 | 10000 48591 05077 17 01 | 10/3 | 11473 56236 88274 29 01 |
| 14 | 10000 61248 13505 87 01 | 3 | 12020 56903 15959 43 01 |
| 41/3 | 10000 77207 23697 67 01 | 8/3 | 12841 90540 23974 21 01 |
| 40/3 | 10000 97332 08000 25 01 | 7/3 | 14151 55609 44598 30 01 |
| 13 | 10001 22713 34757 85 01 | 2 | 16449 34066 84822 64 01 |
| 38/3 | 10001 54728 79053 14 01 | 5/3 | 21235 22968 85758 35 01 |
| 37/3 | 10001 95119 46821 60 01 | 4/3 | 36009 37750 45886 24 01 |
| 12 | 10002 46086 55330 80 01 | 1 | ∞ |
| 35/3 | 10003 10414 39287 00 01 | 2/3 | −24475 80736 23365 82 01 |
| 34/3 | 10003 91627 16273 87 01 | 1/3 | −97336 02483 50782 72 01 |
| 11 | 10004 94188 60411 95 01 | 0 | −50000 00000 00000 00 00 |
| 32/3 | 10006 23757 16251 28 01 | | |