A Calculation of the Number of Lattice Points in the Circle and Sphere

W. Fraser and C. C. Gotlieb

1. Introduction. Let $A_k(x)$ be the number of points $(y_1, y_2, \cdots y_k)$ satisfying

$$y_1^2 + y_2^2 + \cdots y_k^2 \leqq x \tag{1}$$

where $y_1, y_2 \cdots y_k$ are integers (positive, negative or zero). Thus, $A_k(x)$ is the number of lattice points in a k-dimensional hypersphere of radius $x^{1/2}$. This paper describes the calculation of a table of $A_2(x)$ and $A_3(x)$ on an IBM 650 computer.

As a first approximation these are, respectively, the area and volume of the circle and sphere, but the question is how good these approximations are. In general, we are interested in

$$P_k(x) = A_k(x) - V_k(x) \tag{2}$$

where the volume of a sphere of radius $x^{1/2}$ in k-dimensional hyperspace is

$$V(x) = \frac{\Pi^{k/2}\, x^{k/2}}{\Gamma(k/2 + 1)}. \tag{3}$$

$P_2(x)$ has been investigated by many celebrated mathematicians and Wilton [1] gives an account of the early work. More recently there have been theoretical investigations of $P_k(x)$ for higher dimensions, particularly by Walfisz [2], whose notation is being followed here.

We write $P_2(x) = O(x^c)$ to mean, in the usual sense, that there exists K such that

$$| P_2(x) |/x^c \leqq K \quad \text{as } x \to \infty, \text{ and}$$

$$P_2(x) = o(x^c) \quad \text{to mean that}$$

$$P_2(x)/x^c \to 0 \quad \text{as } x \to \infty.$$

Further, after Littlewood, we write $P_2(x) = \Omega(x^c)$ to mean that there exists $K > 0$, and a sequence of values of x tending to infinity, for which

$$| P_2(x) |/x^c \geqq K,$$

that is, the negation of $P_2(x) = o(x^c)$. Gauss observed that

$$P_2(x) = O(x^{1/2})$$

Hua [3] has shown that

$$P_2(x) = O(x^{13/40}),$$

and Van der Corput [4] that

$$P_2(x) = o(x^{1/3})$$

Received August 7, 1961.

TABLE 1
Contributions to $A_3(x)$—First Method

Name	Solutions for:	Conditions	Symmetry Factors		Multiplicity
			Permutation	Reflection	
T	$y_1^2 + y_2^2 + y_3^2 \leqq x$	$y_1 > y_2 > y_3 > 0$	6	8	48
U	$y_1^2 + y_2^2 \leqq x$	$y_1 > y_2 > 0$	6	4	24
V	$2y_1^2 + y_2^2 \leqq x$	$y_1 > y_2 > 0$	3	8	24
W	$y_1^2 + 2y_2^2 \leqq x$	$y_1 > y_2 > 0$	3	8	24
S	$y_1^2 \leqq x$	$y_1 > 0$	3	2	6
Y	$2y_1^2 \leqq x$	$y_1 > 0$	3	4	12
Z	$3y_1^2 \leqq x$	$y_1 > 0$	1	8	8

Hardy has shown that

$$P_2(x) = \Omega(x^{1/4}),$$

so that we may write

$$P_2(x) = O(x^c)$$

where $c \geqq \frac{1}{4}$ and the best value is known to be less than 13/40. It is considered probable that c is arbitrarily close to $\frac{1}{4}$. Besides furnishing numerical values of $A_2(x)$ and $A_3(x)$, one purpose of this tabulation is to examine the consistency of the observed numbers with this conjecture.

Although analytic number theory yields considerable information on the behavior of $P_k(x)$ for $k \geqq 4$, there seems to be little reported on $P_3(x)$. In general

$$P_k(x) = O(x^{(k-1)/2}) \quad \text{and} \quad P_k(x) = \Omega(x^{k/2-1}) \tag{4}$$

so that

$$P_3(x) = O(x^c)$$

where it is known that $c \geqq \frac{1}{2}$ and the best value is equal to or less than 1. These limits, of course, are not very sharp; for $P_2(x)$ we get from (4) only that $\frac{1}{2} \geqq c \geqq 0$. Thus, another purpose of the tabulation is to obtain some information about the behavior of $P_3(x)$.

2. Computing Formulas. There are many summation formulas which can be derived for $A_3(x)$. Essentially they are all modified enumerations where advantage is taken of the symmetries present.

Table 1 shows a decomposition into the terms which contribute to $A_3(x)$. Allowing for the solution at the origin, we then get

$$A_3(x) = 48T + 24(U + V + W) + 6S + 12Y + 8Z + 1^* \tag{5}$$

* As noted by Legendre this same result is reached by noting that $A_3(x)$ is given by the number of terms having coefficients $\leqq x$ in the expansion of $(1 + 2 \sum_i y^{i^2})(1 + 2 \sum_j y^{j^2})(1 + 2 \sum_k y^{k^2})$.

We proceed to find expressions for each of these terms. Let $[\sqrt{N}]$ be the largest integer equal to or less than the square root of N, i.e.,

$$[\sqrt{N}] \leqq \sqrt{N} < [\sqrt{N}] + 1.$$

Then

$$S = [\sqrt{x}]; \qquad Y = [\sqrt{x/2}]; \qquad Z = [\sqrt{x/3}].$$

The other terms in (5) are evaluated by summation formulas. Consider the number of solutions of

$$y_1^2 + y_2^2 \leqq x \quad \text{with} \quad y_1 \neq 0, \quad y_2 \neq 0. \tag{6}$$

For each y_1, the number of permitted values of y_2 is $[\sqrt{x - y_1^2}]$, so that the number of solutions of (6) is given by $\sum_{y_1=1}^{[\sqrt{x}]} [\sqrt{x - y_1^2}]$. But this is equal to

the number of solutions with $y_1 > y_2 > 0$

+ the number of solutions with $y_2 > y_1 > 0$

+ the number of solutions with $y_1 = y_2$.

That is,

$$2U + Y = \sum_{y_1=1}^{[\sqrt{x}]} [\sqrt{x - y_1^2}].$$

A similar argument shows that

$$V + W = \sum_{y_1=1}^{[\sqrt{x/2}]} [\sqrt{x - 2y_1^2}] - [\sqrt{x/3}];$$

again, we have the number of solutions of

$$y_1^2 + y_2^2 + y_3^2 \leqq x \quad \text{for} \quad y_1 > y_2 > 0 = 3T + V + w$$
$$= \sum_{y_1=2}^{[\sqrt{x-1}]} \sum_{y_2=1}^{\min} [\sqrt{x - y_1^2 - y_2^2}]$$

where min is the lesser of $(y_1 - 1)$ and $[\sqrt{x - y_1^2}]$. Thus we finally get

$$A_3(x) = 16 \sum_{y_1=1}^{[\sqrt{x-1}]} \sum_{y_2=1}^{\min} [\sqrt{x - y_1^2 - y_2^2}] + 8 \sum_{y_1=1}^{[\sqrt{x/2}]} [\sqrt{x - 2y_1^2}] + 12 \sum_{y_1=1}^{[\sqrt{x}]} [\sqrt{x - y_1^2}] + 6[\sqrt{x}] + 1. \tag{7}$$

Further, it is easily seen that

$$A_2(x) = 4 \sum_{y_1=1}^{[\sqrt{x}]} [\sqrt{x - y_1^2}] + 4[\sqrt{x}] + 1. \tag{8}$$

These equations, (7) and (8), were the two actually used the first time the table was computed. If it had been practical to compute in successive values of x it probably would have been best to use a difference formula to find the contributions from successive spherical shells. Actually it was decided to compute in equal intervals of $x^{1/2}$ for a while, and then to skip to large arguments so that the asymptotic values could be examined. The IBM 650 computer used for the calculation was equipped with core storage and index registers. Although the full program required only 168 instructions and the inner loop was completely in the 60-word core storage, it took about 10 hours to compute $A_3(10^6)$, the largest value for which equation (7) was used.

A second and independent program was written for computing $A_2(x)$ and $A_3(x)$. This program was based on the equations:

$$
\begin{aligned}
A_3(x) &= 1 + 6\,[\sqrt{x}] + 12\left[\sqrt{\frac{x}{2}}\right] + 8\left[\sqrt{\frac{x}{3}}\right] + 24\sum_{y_1=1}^{L_1}([\sqrt{x - y_1^2}] - y_1) \\
&\quad + 24\sum_{y_1=1}^{L_2}([\sqrt{x - 2y_1^2}] - y_1) + 24\sum_{y_1=1}^{L_3}\left(\left[\sqrt{\frac{x - y_1^2}{2}}\right] - y_1\right) \qquad (9)\\
&\quad + 48\sum_{y_1=1}^{L_4}\sum_{y_2=y_1+1}^{L_5}([\sqrt{x - y_1^2 - y_2^2}] - y_2),
\end{aligned}
$$

where

L_1 is the largest integer, y_1 satisfying $y_1^2 + (y_1 + 1)^2 \leqq x$

L_2 is the largest integer, y_1 satisfying $2y_1^2 + (y_1 + 1)^2 \leqq x$

L_3 is the largest integer, y_1 satisfying $y_1^2 + 2(y_1 + 1)^2 \leqq x$

L_4 is the largest integer, y_1 satisfying $y_1^2 + (y_1 + 1)^2 + (y_1 + 2)^2 \leqq x$

L_5 is the largest integer, $y_2(y_1)$ satisfying $y_1^2 + y_2^2 + (y_1 + 1)^2 \leqq x$;

$$
A_2(x) = 1 + 4[\sqrt{x}] + 4[\sqrt{x/2}] + 8\sum_{y_1=1}^{L_0}([\sqrt{x - y_1^2}] - y_1) \qquad (10)
$$

where L_0 is the largest integer, y_1 satisfying $y_1^2 + (y_1 + 1)^2 \leqq x$. These equations can be obtained by counting the lattice points according to the scheme illustrated in Figure 1 and Table 2.

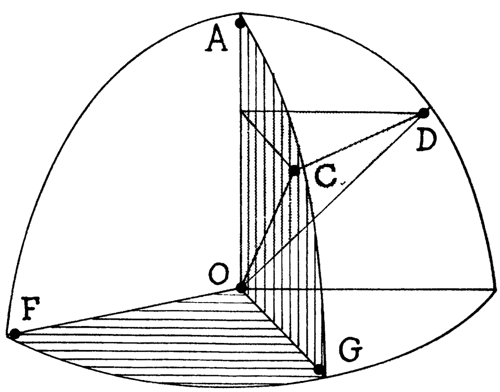

FIG. 1. Decomposition of $A_3(x)$—second method.

TABLE 2
Contributions to $A_3(x)$—Second Method

Source	Multiplicity	Number of Points
Point O	1	1
Line OF	6	$[\sqrt{x}]$
Line OG	12	$[\sqrt{x/2}]$
Line OC	8	$[\sqrt{x/3}]$
Plane OFG	24	$\sum([\sqrt{x - y_1^2}] - y_1)$
Plane OCA	24	$\sum([\sqrt{x - 2y_1^2}] - y_1)$
Plane OCG	24	$\sum\left(\left[\sqrt{\frac{x - y_1^2}{2}}\right] - y_1\right)$
Volume OACD	48	$\sum\sum([\sqrt{x - y_1^2 - y_2^2}] - y_2)$

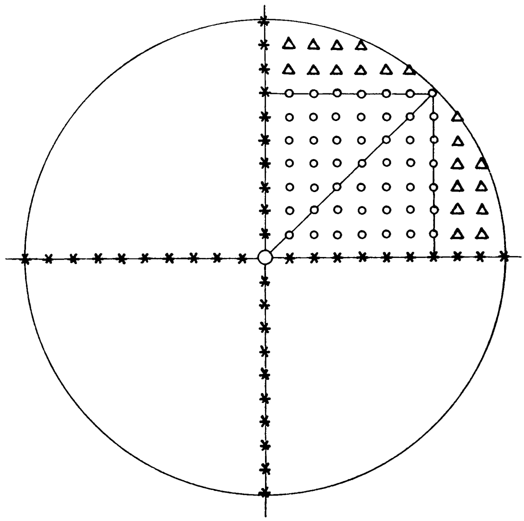

FIG. 2. Decomposition of $A_2(x)$—Gauss method.

This second program was faster than the first, and it was possible to extend the calculation to $x = 3.24 \times 10^6$, which value took about $1\frac{1}{2}$ hours to compute. For this value, $A_3(x)$ exceeds the 10 digits of the storage positions, but the individual sums contributing to the result are still within single-precision range.

Since the calculation was done it has been realized that there are formulas which might be even more efficient for computing. These result from the following equation, noted by Gauss:

$$(11) \qquad A_2(x) = 1 + 4[\sqrt{x}] + 4[\sqrt{x/2}]^2 + 8 \sum_{y_1 = [\sqrt{x/2}]+1}^{[\sqrt{x}]} [\sqrt{x - y_1^2}].$$

The terms contributing to $A_2(x)$ are shown in Figure 2, and it is seen that equation (11) is superior to equations (8) and (10) for computing because the contribution given by $4[\sqrt{x/2}]^2$ has been removed from the summation. When equation (11)

TABLE 3

Computed Results

$X^{1/2}$	$A_3(X)$	$\vert P_3(X)\vert$	$A_2(X)$	$\vert P_2(X)\vert$
1	7	3	5	2
2	33	1	13	0
3	123	10	29	1
4	257	11	49	1
5	515	9	81	2
6	925	20	113	0
7	1419	18	149	5
8	2109	36	197	4
9	3071	17	253	1
10	4169	20	317	3
11	5575		377	3
12	7153	85	441	11
13	9171	32	529	2
14	11513	19	613	3
15	14147	10	709	2
16	17077	80	797	7
17	20479	101	901	7
18	24405	24	1009	9
19	28671	60	1129	5
20	33401	109	1257	0
21	38911	119	1373	12
22	44473	129	1517	4
23	50883	82	1653	9
24	57777	129	1793	17
25	65267	183	1961	2
26	73525	97	2121	3
27	82519	71	2289	1
28	91965	13	2453	10
29	101943	217	2629	13
30	113081	16	2821	6
31	124487	301	3001	18
32	137065	193	3209	8
33	150555	22	3409	12
34	164517	119	3625	7
35	179579	15	3853	5
36	195269	163	4053	19
37	212095	80	4293	8
38	229549	298	4513	23
39	248439	36	4777	1
40	267761	322	5025	2
41	288359	337	5261	20
42	310177	162	5525	17
43	332779	259	5789	20
44	356637	181	6077	5
45	381915	211	6361	1
46	407597	123	6625	23
47	434551	342	6921	19
48	462781	466	7213	25
49	492567	240	7525	18

TABLE 3—*Continued*

$X^{1/2}$	$A_3(X)$	$\mid P_3(X)\mid$	$A_2(X)$	$\mid P_2(X)\mid$
50	523305	294	7845	9
55	696507	403	9477	26
60	904089	690	11289	21
65	1149651	696	13273	0
70	1436385	370	15373	21
75	1767063	83	17665	6
80	2143641	1020	20081	25
85	2571711	730	22701	3
90	3053617	11	25445	2
95	3590863	501	28345	8
100	4187857	933	31417	1
105	4849327	279	34621	15
110	5574721	559	37981	32
115	6370351	275	41545	3
120	7236577	1652	45225	14
125	8180887	344	49077	10
130	9201625	1147	53077	16
135	10305407	588	57209	47
140	11492081	1959	61529	46
145	12768503	1548	66045	7
150	14137637	470	70681	5
155	15598031	500	75465	12
160	17155325	1960	80381	44
165	18817007	438	85501	29
170	20578325	1201	90785	7
175	22448927	371	96209	2
180	24427317	1707	101765	23
185	26520663	1186	107501	20
190	28729653	1259	113369	42
195	31058271	1085	119433	26
200	33507885	2437	125629	35
300	113094545	2791	282697	46
400	268077737	4836	502625	30
500	523592077	6699	785349	49
600	904769241	9443	1130913	60
700	1436743985	11055	1539297	83
800	2144654669	5916	2010573	46
900	3053616505	11554	2544569	121
1000	4188781437	8768	3141549	44
1200	7238202017	27457	4523793	100
1400	11494026189	14133	6157477	145
1600	17157266213	18466	8042349	128
1800	24428980617	43857	10178545	215

is integrated plane-by-plane there results

$$A_3(x) = 1 + 6[\sqrt{x}] + 12[\sqrt{x/2}] + 24 \sum_{y_1=[\sqrt{x/2}]+1}^{[\sqrt{x}]} [\sqrt{x - y_1^2}]$$

$$(12) \qquad + 8 \sum_{y_1=1}^{[\sqrt{x-1}]} \left[\sqrt{\frac{x - y_1^2}{2}}\right] + 16 \sum_{y_1=1}^{[\sqrt{x-1}]} \sum_{y_2=\left[\sqrt{\frac{x-y_1^2}{2}}+1\right]}^{[\sqrt{x-y_1^2}]} [\sqrt{x - y_1^2 - y_2^2}].$$

Still other summation formulas can be derived and it is likely that the most efficient one will depend on the computer being used. A method of decomposition for finding $A_k(x)$ which is best for $k = 2$ is not necessarily best for higher values of k.

3. Results. A partial table of computed results is shown in Table 3. Shown there are $x^{1/2}$, $A_3(x)$, $|P_3(x)|$, $A_2(x)$ and $|P_2(x)|$ for the following range:

$$x^{1/2} = 1(1)\ 50(5)\ 200(100)\ 1000(200)\ 1800.$$

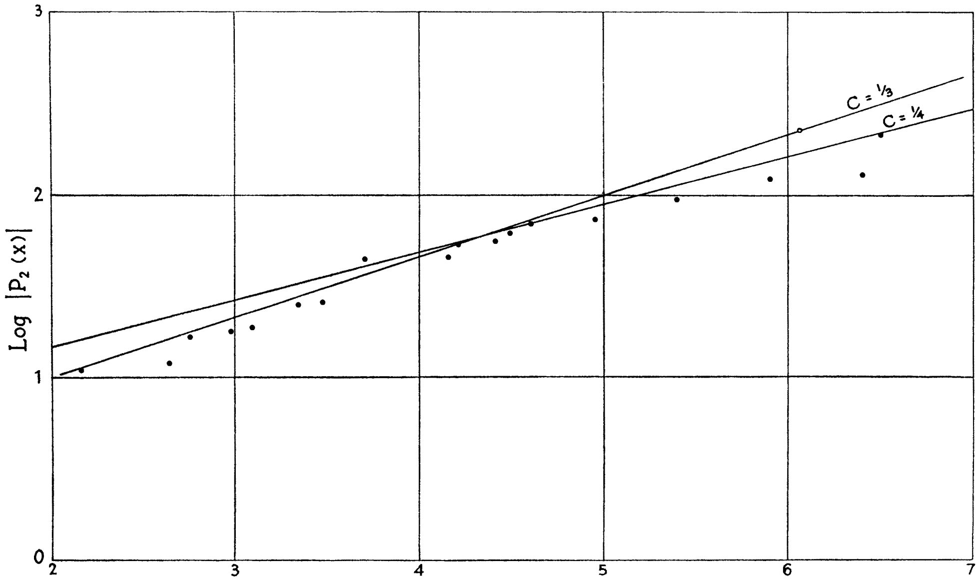

FIG. 3. Log X.

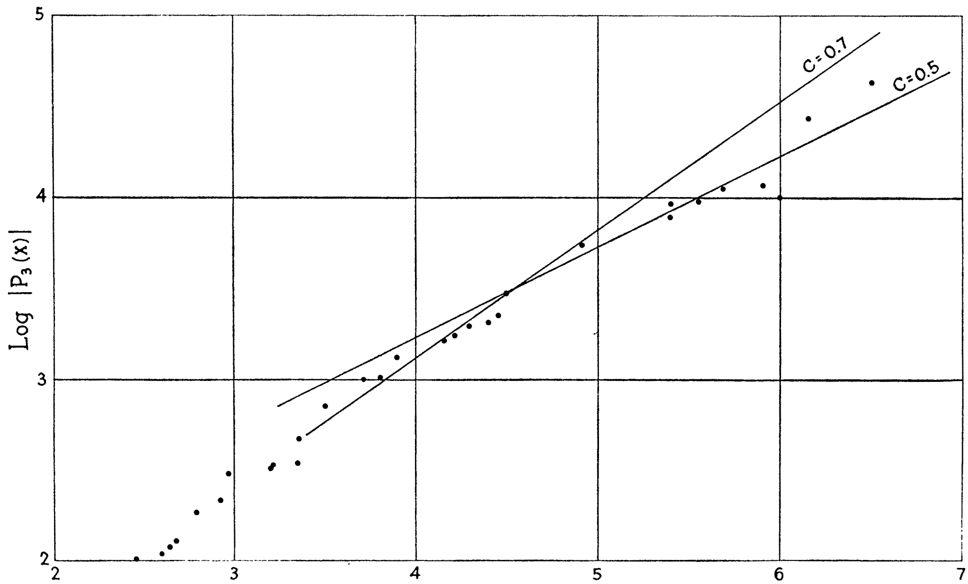

FIG. 4. Log X.

All of these entries except those for $x^{1/2} = 1200, 1400, 1600$ and 1800 were computed by the two independent programs. The results agree with those given in the short table of reference [5]. Calculations were actually made for about 250 arguments in the range $x^{1/2} = 1$ to 2000.

Because it is known that $P_2(x) = \Omega(x^{1/4})$ the points of greatest interest in investigating the asymptotic behavior of $P_2(x)$ are those local maxima, M_i of $| P_2(x) |$, satisfying $| P_2(M_i) | > | P_2(x) |$ for all $x < M_i$. From the tabulated results it is only possible to obtain an estimate of these M_i and the corresponding values of $| P_2(M_i) |$.

Figure 3 shows a graph of $\log | P_2(x) |$ versus $\log x$ for those computed values of x where $\log | P_2(x) |$ is larger than any preceding value. We have drawn two lines on this graph, one with slope 1/3, and the other with slope 1/4. The line with slope 1/3 looks too steep, that is, one feels that the points will continue to lie more and more below the line. The line with slope 1/4 looks reasonable, from which one can conclude that the conjecture that c is arbitrarily close to 1/4 is not inconsistent with the observed results. However some unpublished computations by Harry Mitchell of the Lockheed Missiles and Space Corporation show that for some x between 10^6 and 10^{10} the values of $| P_2(x) |/x^{1/4}$ grow very remarkably.

Figure 4 shows a graph of $\log | P_3(x) |$ versus $\log x$, for those computed values of x where $\log | P_3(x) |$ is larger than any preceding value. Two lines, with slopes 0.5 and 0.7 have been drawn. It looks as if the points will, in the main, continue to lie between these lines, from which we conjecture that

$$| P_3(x) | = O(x^c)$$

where $0.5 \leqq c \leqq 0.7$.

We would like to thank D. B. Scott, University of Toronto, whose investigations into close packing of spheres first suggested this calculation, and G. Forsythe, Stanford University, with whom we have had helpful discussions.

Defence Research Board
University of Toronto

Computation Center
University of Toronto

1. J. R. Wilton, "The lattice points of a cirlce: an historical account of the problem," *Messenger of Mathematics*, v. 58, 1929, p. 67–80.
2. A. Walfisz, *Gitterpunkte in Mehrdimensionalen Kugeln*, Monografie Matematyczne, v. 33, Panstwowe Wydawnictwo Naukowe, Warsaw, 1957. (Reviewed in *Mathematical Reviews*, v. 20, n. 6, 1959, p. 3826.)
3. L. K. Hua, "The lattice-points in a circle," *Quart. J. Math., Oxford Ser.* 13, 1942, p. 18–29.
4. E. Landau, *Vorlesungen üeber Zahlentheorie*, Von S. Hirzel, Leipzig, 1927, p. 264.
5. Hilbert & Cohn-Vossen, *Anschauliche Geometrie*, Dover, New York, 1944. p. 29.