A Method for Computing the Circular Coverage Function

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1. Introduction. In this paper an efficient method is described for the numerical evaluation, with a high-speed digital computer, of a special case of the integral of an uncorrelated bivariate Gaussian distribution centered at the origin over the area of an arbitrarily placed circle in the plane. This function, popularly known as the circular coverage function or as the non-central chi-square distribution for two degrees of freedom*, can be written as

$$(1) \qquad P(R, D) \equiv \frac{1}{2\pi\sigma_x \sigma_y} \iint_S \exp\left\{ -\frac{1}{2}\left[\left(\frac{x}{\sigma_x}\right)^2 + \left(\frac{y}{\sigma_y}\right)^2\right]\right\} dxdy,$$

where S is the circle: $(x - h)^2 + (y - k)^2 = (\sigma R)^2$, where $\sigma_x = \sigma_y = \sigma$, and σD is the radial distance from the origin to the center (h, k) of the circle of integration, S. Because of the equivalence mentioned above, a great deal of published literature applies. The papers [13], [15], suggested by the referee, list a large number of such references.

The average computing time for the calculation of the integral in equation (1) to six decimal digits, by the method of this paper, is six milliseconds on the IBM 7090 and ten milliseconds on NORC. An extensive inverse table, which is described in the last section of this paper and which is given in [4], has been computed with R as a function of P and D. A condensed version, Table 1, is presented herein.

In the general case [3], [11] suppose the uncorrelated bivariate Gaussian distribution centered at the origin of an Oxy Cartesian coordinate system has standard deviations σ_x, σ_y along the x and y axes respectively, and that the integral of this function is to be evaluated over a circle of radius $\bar{R}$ with center at (h, k). Then the probability, P, can be written in polar coordinates accordingly:

$$(2) \qquad \begin{aligned} P\left(\frac{\bar{R}}{\sigma_x}, \frac{\bar{R}}{\sigma_y}, \frac{h}{\sigma_x}, \frac{k}{\sigma_y}\right) &= \frac{1}{2\pi\sigma_x \sigma_y} \int_0^{\bar{R}} \int_0^{2\pi} \\ &\cdot \exp\left\{ -\frac{1}{2}\left[\left(\frac{h + r\cos\theta}{\sigma_x}\right)^2 + \left(\frac{k + r\sin\theta}{\sigma_y}\right)^2\right]\right\} rdrd\theta, \end{aligned}$$

where $x - h = r\cos\theta$, $y - k = r\sin\theta$, $0 \leqq r \leqq \bar{R}$, $0 \leqq \theta \leqq 2\pi$.

If

$$h = k = 0,$$

a special case identified as the $V(K, c)$ or elliptical normal probability function (sometimes known by other titles, for example, the generalized circular error function) [4], [5], [6], [10], [14], [15], [16], [18] follows, i.e.,

$$(3) \qquad P\left(\frac{\bar{R}}{\sigma_x}, \frac{\bar{R}}{\sigma_y}, 0, 0\right) \equiv V(K, c) = \frac{1}{c}\int_0^K \exp\left(-\frac{B}{2}r^2\right) I_0\left(\frac{Ar^2}{2}\right) rdr,$$

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* The equivalence between the function $P(R, D)$ of equation (1) and the non-central chi-square distribution is evident from equation (2) in [13].

where

$$0 \leqq c \equiv \frac{\sigma_y}{\sigma_x} \leqq 1, \qquad K \equiv \bar{R}/\sigma_x\,, \qquad A \equiv \frac{1-c^2}{2c^2}\,, \qquad B \equiv \frac{1+c^2}{2c^2}\,.$$

$I_0(x)$ is the modified Bessel function of the first kind of order zero, [8]. Equation (3) is derived by setting $h = k = 0$ in equation (2), by using the trigonometric identity $1(\pm) \cos 2\theta = 2\begin{pmatrix}\cos^2\theta \\ \sin^2\theta\end{pmatrix}$, and by introducing an integral expression for $I_0(x)$ which is given by

$$I_0(x) = \frac{1}{\pi}\int_0^\pi \exp(-x\cos\theta)\,d\theta. \tag{4}$$

Equation (4) can be derived from Example 1 (ii), page 62, in [8].

If
$$\sigma_x = \sigma_y = \sigma,$$
in equation (2), the distribution is circular normal. In this case, in which h and k are arbitrary, the center of the circle of integration can always be taken as offset a distance of σD from the origin along the positive x axis by simply introducing a rotation of axes through the angle arc tan $\left(\frac{k}{h}\right)$. Moreover, by introducing the integral expression for $I_0(x)$ as given by equation (4), the circular coverage function, $P(R, D)$, [1], [4], [6], [7], [9], [12], [13], [14], [17], is obtained from equation (2), i.e.,

$$P\left(\frac{\bar{R}}{\sigma_x}, \frac{\bar{R}}{\sigma_x}, \frac{h}{\sigma_x}, \frac{k}{\sigma_x}\right) \equiv P(R, D) = \exp(-D^2/2)\int_0^R \exp(-r^2/2) I_0(rD) r\,dr, \tag{5}$$

where $R \equiv \bar{R}/\sigma_x$, $D^2 \equiv (h^2 + k^2)/\sigma_x^2$.

The function $\partial P(R, D)/\partial R$ is required for computing the inverse function, $R(P, D)$, by the Newton-Raphson procedure (Appendix C, [4]) and is also of use in computing $P(R, D)$ itself (see equation (9)). This function is obtained straightforwardly from equation (5) as

$$\frac{\partial P}{\partial R} = R \exp\left(-\frac{R^2 + D^2}{2}\right) I_0(RD). \tag{6}$$

It is apparent by comparing equations (6), (9) that $\partial P/\partial R$ can be computed simultaneously with $P(R, D)$.

In a previous paper, [18], a very efficient computing method was described for calculation of the $V(K, c)$ function. The success of the method warranted consideration of extending the technique to the $P(R, D)$ function. This is not as straightforward as for $V(K, c)$; nevertheless, it is easily possible because of the existence of a simple functional relationship, equation (9), between $P(R, D)$ and $V(K, c)$.

2. The Relationship between P(R, D) and V(K, c). The relationship between P and V can be derived by utilizing two preliminary results which are given by Fettis, in terms of $q \equiv 1 - P$, in equations (I-35) and (I-44) in [6]. They can be stated in terms of P as:

$$P(R,D) - P(D,R) = \pm V\left(|R-D|, \frac{|R-D|}{R+D}\right) \quad \begin{matrix}(+) \text{ if } R > D \\ (-) \text{ if } R < D,\end{matrix} \tag{7}$$

$$P(R,D) + P(D,R) = 1 - \exp\left(-\frac{R^2+D^2}{2}\right) I_0(RD). \tag{8}$$

Equation (8) is easily derived. The origin of equation (7) is not known to the authors. The referee has pointed out that a geometrical proof was given by Dr. David C. Kleinecke of the University of California in 1955. (See also paper I of [15], page 613). Mr. Fettis has kindly placed at the disposal of the authors some correspondence which indicates that the relationship was given in a Sandia Corporation working paper in 1952, and that it was believed to have been originally derived in a British publication by using power series.

It follows by adding the corresponding sides of equations (7) and (8) that*

$$P(R,D) = \frac{1}{2}\left[1 - \exp\left(-\frac{R^2+D^2}{2}\right) I_0(RD) \pm V\left(|R-D|, \frac{|R-D|}{R+D}\right)\right] \quad \begin{matrix}(+) \text{ if } R > D \\ (-) \text{ if } R < D.\end{matrix} \tag{9}$$

Thus, the $P(R, D)$ function is computable at virtually the same speed as $V(K, c)$, since the second term in the brackets turns out to be a by-product of the recurrence relations which are used to compute V in the last term. Consequently, if there exists a satisfactory computing program for the V function, a computing program of equal merit can be realized for the $P(R, D)$ function.

3. Recurrence Relations. The V function that appears as the last term of equation (9) is identified with equation (3) by setting

$$K = |R-D|, \qquad c = |R-D|/(R+D).$$

It follows that

$$A = \frac{2RD}{(R-D)^2}, \qquad B = \frac{R^2+D^2}{(R-D)^2},$$

where it is assumed $R \neq D$. If $R = D$, then, from equation (7), $V\left(|R-D|, \frac{|R-D|}{R+D}\right)$ vanishes and $P(R, D)$ is given by the first two terms of equation (9).

The two series representations for $V\left(|R-D|, \frac{|R-D|}{R+D}\right)$ from which the basic recurrence relations are derived are given by:

* Guenther recently (see equation (2) in [9]) derived an equation for $P(R, D)$ in terms of $I_0(x)$ and the incomplete gamma function, which can be shown to be equivalent to equation (9) of the present paper. However, he did not exploit his relationship from the point of view of developing an efficient program for a high-speed digital computer.

$$V\left(|R - D|, \frac{|R - D|}{R + D}\right) = \frac{|R^2 - D^2|}{RD} \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right)^2 \cdot \int_0^{RD/2} \exp\left(-\frac{R^2 + D^2}{RD} w\right) w^{2n} \, dw \equiv \sum_{n=0}^{\infty} T_{2n}, \tag{10}$$

$$V\left(|R - D|, \frac{|R - D|}{R + D}\right) = 1 - \frac{|R^2 - D^2|}{4RD\sqrt{\pi}} \sum_{n=0}^{N} \frac{[(2n)!]^2}{2^{4n}(n!)^3} \cdot \int_{2RD}^{\infty} \exp\left[-\frac{(R - D)^2}{4RD} w\right] w^{-\left(\frac{2n+1}{2}\right)} \, dw = 1 - \sum_{n=0}^{N} M_{2n+1}. \tag{11}$$

The detailed derivations of equations (10), (11) are given in [4]. Briefly, to obtain equation (10), introduce a variable of integration transformation

$$w = Ar^2/4 \tag{12}$$

into the integral of equation (3), then replace $I_0(2w)$ by its Taylor series expansion (see page 14, [8]),

$$I_0(2w) = \sum_{n=0}^{\infty} \left(\frac{1}{n!}\right)^2 \left(\frac{2w}{2}\right)^{2n}, \tag{13}$$

which is convergent for all values of w, and subsequently reverse the order of integration and summation, which can be justified by application of the Weierstrass "M" test. In order to derive equation (11) introduce a variable of integration transformation

$$w = Ar^2 \tag{14}$$

into the integral of equation (3) and use the fact that

$$\frac{1}{2Ac} \int_0^{\infty} \exp\left(-\frac{Bw}{2A}\right) I_0\left(\frac{w}{2}\right) dw = 1, \tag{15}$$

(See page 76, [8]). In the resulting integral expression, call it J, with upper and lower limits of integration of infinity and AK^2 respectively, replace $I_0\left(\frac{w}{2}\right)$ by its asymptotic expansion (see page 58, [8]), i.e.,

$$I_0\left(\frac{w}{2}\right) \approx \frac{\exp(w/2)}{\sqrt{2\pi(w/2)}} \sum_{n=0}^{N} \frac{[(2n)!]^2}{2^{4n}(n!)^3} (2w/2)^{-n}, \tag{16}$$

which is valid for sufficiently large w and finite N; subsequently interchange the order of integration and summation. The interchange is justified for all values of $(2RD)$ for which equation (16) is valid because of the existence of the integral J (see page 17, [2]).

The substitution of equations (13), (16) into equation (6) gives analogous series representations for $\partial P/\partial R$, i.e.,

$$\frac{\partial P}{\partial R} = R \sum_{n=0}^{\infty} \bar{S}_{2n}, \tag{17}$$

$$\frac{\partial P}{\partial R} \approx R\left(\frac{1}{4RD} \sum_{n=0}^{N} \bar{X}_{2n+1}\right), \tag{18}$$

where

$$\bar{S}_{2n} \equiv \exp\left(-\frac{R^2 + D^2}{2}\right)\left(\frac{1}{n!}\right)^2\left(\frac{RD}{2}\right)^{2n}, \qquad n \geqq 0, \tag{19}$$

$$\bar{X}_{2n+1} \equiv \frac{2}{\sqrt{\pi}} \exp\left[-\frac{(R - D)^2}{2}\right]\frac{[(2n)!]^2}{2^{4n}(n!)^3}(2RD)^{-\left(\frac{2n-1}{2}\right)}, \qquad n \geqq 0, \tag{20}$$

following the notation of [4], in which there are slight distinctions between $\bar{S}_{2n}$, $\bar{X}_{2n+1}$, $\bar{Y}_{2n-1}$, and the corresponding unbarred variables used with $V(K, c)$ and $\partial V/\partial K$.

Thus two schemes are used to compute P. If

$$2RD \leqq M \quad (M \text{ is a positive constant}), \tag{21}$$

then with reference to equations (10) and (17)

$$\begin{aligned} T_{2n} &= \left(\frac{2n - 1}{2n}\right)\left(\frac{2RD}{R^2 + D^2}\right)^2 T_{2n-2} \\ &\quad - \frac{|R^2 - D^2|}{R^2 + D^2}\left(1 + \frac{4n}{R^2 + D^2}\right)\bar{S}_{2n}, \qquad n \geqq 1, \end{aligned} \tag{22}$$

$$\bar{S}_{2n} = \left(\frac{RD}{2n}\right)^2 \bar{S}_{2n-2}, \qquad n \geqq 1, \tag{23}$$

where the necessary initial terms are given by

$$T_0 = \frac{|R^2 - D^2|}{R^2 + D^2}\left[1 - \exp\left(-\frac{R^2 + D^2}{2}\right)\right] = \frac{|R^2 - D^2|}{R^2 + D^2}(1 - \bar{S}_0), \tag{24}$$

$$\bar{S}_0 = \exp\left(-\frac{R^2 + D^2}{2}\right). \tag{25}$$

The following brief comments are made on the derivation of recurrence relations (22) and (23). Fuller details are given in [4]. From equations (13) and (19), $\bar{S}_{2n}$ is the general term in the series obtained by multiplying every term of the Taylor series for $I_0(RD)$ by $\exp[-(R^2 + D^2)/2]$, and equations (23) and (25) are obtained immediately. If T_{2n} is regarded as defined by equation (10), two successive integrations by parts give T_{2n} in terms of T_{2n-2}, R, D, and n, after which the term not containing T_{2n-2} can be written more concisely in terms of $\bar{S}_{2n}$, and equation (22) is the result.

These basic recurrence relations are cycled until

$$T_{2n} < \epsilon, \qquad \bar{S}_{2n} < \epsilon, \qquad (\epsilon > 0). \tag{26}$$

Then P and $\partial P/\partial R$ are given correctly to at least $(|\log_{10} \epsilon| - 1)$ decimal digits by

$$P(R, D) \approx \frac{1}{2}\left[1 - \sum_{n=0}^{N'} \bar{S}_{2n} \pm \sum_{n=0}^{N'} T_{2n}\right] \qquad \begin{array}{l}(+) \text{ if } R > D \\ (-) \text{ if } R < D,\end{array} \tag{27}$$

$$\frac{\partial P}{\partial R} \approx R \sum_{n=0}^{N'} \bar{S}_{2n}. \tag{28}$$

If it is assumed that

$$2RD > M, \tag{29}$$

then with reference to equations (11) and (18)

$$M_{2n+1} = \frac{|R^2 - D^2|}{4RD} \bar{Y}_{2n-1} - \frac{(R-D)^2}{4RD}\left(\frac{2n-1}{2n}\right) M_{2n-1}, \qquad n \geqq 1, \tag{30}$$

$$\bar{Y}_{2n-1} = \frac{1}{4RD}\left(\frac{2n-1}{2n}\right)\bar{X}_{2n-1}, \qquad n \geqq 1, \tag{31}$$

$$\bar{X}_{2n+1} = (2n-1)\bar{Y}_{2n-1}, \qquad n \geqq 1, \tag{32}$$

where the initial terms are given by

$$\begin{aligned} M_1 &= \frac{1}{\sqrt{2RD}}\left(\frac{R+D}{\sqrt{2}}\right)\frac{2}{\sqrt{\pi}}\int_{\frac{|R-D|}{\sqrt{2}}}^{\infty} \exp(-y^2)\, dy \\ &= \frac{1}{\sqrt{2RD}}\left(\frac{R+D}{\sqrt{2}}\right)\left[1 - \operatorname{Erf}\left(\frac{|R-D|}{\sqrt{2}}\right)\right], \end{aligned} \tag{33}$$

$$\bar{X}_1 = \sqrt{2RD}\,\frac{2}{\sqrt{\pi}} \exp\left[-\frac{(R-D)^2}{2}\right]. \tag{34}$$

The following brief comments are made on the derivation of equations (30) to (33). Fuller details are given in [4]. From equations (16) and (20), $\bar{X}_{2n+1}$ is the general term in the expansion obtained by multiplying every term of the asymptotic expansion of $I_0(RD)$ by $4RD \exp[-(R^2 + D^2)/2]$. Equations (31) and (32), which together form a recurrence relation generating $\bar{X}_{2n+1}$, are obtained immediately, the introduction of the variable $\bar{Y}_{2n-1}$ leading to a computationally efficient algorithm for the simultaneous evaluation of the last two terms in equation (9). If M_{2n+1} is regarded as defined by equation (11), an integration by parts gives M_{2n+1} in terms of M_{2n-1}, R, D, and n, after which the term not containing M_{2n-1} can be written more concisely in terms of $\bar{Y}_{2n-1}$, and recurrence relation (30) is the result. M_1, originally obtained by putting $n = 0$ in the definition of M_{2n+1}, is expressed in equation (33) in terms of the error function (see [3], equations (6)) by a transformation in which y is $(\frac{1}{2})|R - D|\sqrt{w/(RD)}$.

These basic recurrence relations are cycled until

$$M_{2n+1} < \epsilon, \qquad \bar{X}_{2n+1} < \epsilon, \qquad (\epsilon > 0). \tag{35}$$

Then $P(R, D)$ and $\partial P/\partial R$ are given correctly to $(|\log_{10} \epsilon| - 1)$ decimal digits by

$$P(R, D) \approx \frac{1}{2}\left[1 - \frac{1}{4RD}\sum_{n=0}^{N} \bar{X}_{2n+1} \pm \left(1 - \sum_{n=0}^{N} M_{2n+1}\right)\right] \quad \begin{matrix} (+) \text{ if } R > D \\ (-) \text{ if } R < D, \end{matrix} \tag{36}$$

$$\frac{\partial P}{\partial R} \approx R\left[\frac{1}{4RD}\sum_{n=0}^{N} \bar{X}_{2n+1}\right]. \tag{37}$$

The determination of the constant M is discussed in Appendix A of [4]. If the constants M and ϵ were chosen such that

$$M = 30, \qquad \epsilon = 10^{-8}, \tag{38}$$

then sufficient tests were made on the results to assure seven-decimal digit accuracy in the values of P and $\partial P/\partial R$ for all values of R and D. The tests are described in [4]. The maximum number of terms, N', required for seven-decimal digit accuracy in either series that occurs in equation (27) was twenty for $0 < R \leqq 126$, $0 \leqq D \leqq 120$.

4. Table Computation—Discussion of Results. The extensive inverse table, mentioned in the introduction, has R tabulated as a function of P and D for the

TABLE 1
Inverse $P(R, D)$ Table, $R = R(P, D)$

P \ D	0.1	0.5	1.0	1.5	2.0	3.0	4.0	5.0
.01	0.142132	0.150917	0.181965	0.247976	0.377894	0.973968	1.857355	2.807007
.05	0.321093	0.340911	0.410355	0.552995	0.803492	1.589932	2.514287	3.475659
.10	0.460192	0.488541	0.586808	0.780875	1.090931	1.931431	2.867729	3.833372
.15	0.571548	0.606683	0.727145	0.956651	1.299471	2.164629	3.107065	4.075094
.20	0.669719	0.710800	0.850071	1.106744	1.470965	2.351156	3.297689	4.267393
.25	0.760426	0.806964	0.962923	1.241576	1.621141	2.511865	3.461479	4.432486
.30	0.846714	0.898407	1.069594	1.366651	1.757905	2.656649	3.608743	4.580828
.35	0.930528	0.987190	1.172547	1.485396	1.885955	2.791156	3.745340	4.718356
.40	1.013296	1.074827	1.273564	1.600226	2.008448	2.919061	3.875068	4.848912
.45	1.096204	1.162568	1.374100	1.713036	2.127745	3.043037	4.000676	4.975274
.50	1.180355	1.251580	1.475479	1.825472	2.245802	3.165246	4.124378	5.099676
.55	1.266891	1.343064	1.579042	1.939121	2.364426	3.287634	4.248157	5.224119
.60	1.357113	1.438388	1.686286	2.055680	2.485472	3.412162	4.374006	5.350606
.65	1.452637	1.539246	1.799042	2.177146	2.611062	3.541034	4.504154	5.481380
.70	1.555634	1.647914	1.919739	2.306101	2.743883	3.677012	4.641388	5.619238
.75	1.669270	1.767705	2.051892	2.446209	2.887695	3.823927	4.789566	5.768053
.80	1.798604	1.903913	2.201075	2.603222	3.048351	3.987718	4.954663	5.933817
.85	1.952745	2.066052	2.377281	2.787369	3.236215	4.178871	5.147218	6.127099
.90	2.151322	2.274618	2.601947	3.020515	3.473382	4.419704	5.389656	6.370384
.95	2.453851	2.591661	2.939763	3.368463	3.826253	4.777225	5.749279	6.731139
.97	2.654829	2.801806	3.161592	3.595668	4.056141	5.009727	5.982997	6.965523
.99	3.042407	3.205999	3.584494	4.026818	4.491533	5.449368	6.424667	7.408327
.995	3.263342	3.435790	3.823110	4.269216	4.735933	5.695826	6.672133	7.656366
.999	3.726147	3.915765	4.318250	4.770776	5.240984	6.204548	7.182694	8.167991
.9999	4.302554	4.511127	4.927840	5.386401	5.860000	6.827233	7.807274	8.793692
.99999	4.810368	5.033640	5.459903	5.922582	6.398559	7.368429	8.349868	9.337129
.999999	5.269458	5.504595	5.937784	6.403513	6.881283	7.853179	8.835714	9.823646

P \ D	6.0	8.0	10.0	20.0	30.0	50.0	80.0	120.0
.01	3.778556	5.747335	7.730490	17.70022	27.69100	47.68389	77.67999	117.6779
.05	4.452164	6.424982	8.409712	18.38123	28.37229	48.36531	78.36146	118.3593
.10	4.811875	6.786445	8.771899	18.74428	28.73548	48.72858	78.72475	118.7226
.15	5.054765	7.030393	9.016299	18.98923	28.98053	48.97367	78.96986	118.9678
.20	5.247904	7.224314	9.210559	19.18391	29.17528	49.16846	79.16466	119.1626
.25	5.413665	7.390705	9.377228	19.35094	29.34237	49.33558	79.33179	119.3297
.30	5.562570	7.540148	9.526912	19.50093	29.49241	49.48565	79.48187	119.4798
.35	5.700590	7.678645	9.665623	19.63992	29.63145	49.62472	79.62094	119.6189
.40	5.831589	7.810077	9.797253	19.77181	29.76339	49.75668	79.75291	119.7508
.45	5.958359	7.937251	9.924613	19.89941	29.89104	49.88435	79.88059	119.8785
.50	6.083144	8.062420	10.04996	20.02499	30.01667	50.01000	80.00625	120.0042
.55	6.207953	8.187598	10.17531	20.15058	30.14229	50.13565	80.13191	120.1298
.60	6.334797	8.314803	10.30269	20.27819	30.26994	50.26332	80.25959	120.2575
.65	6.465923	8.446290	10.43434	20.41008	30.40188	50.39528	80.39156	120.3895
.70	6.604135	8.584868	10.57310	20.54907	30.54092	50.53435	80.53063	120.5286
.75	6.753314	8.734427	10.72284	20.69907	30.69097	50.68442	80.68071	120.6786
.80	6.919464	8.900983	10.88959	20.86610	30.85806	50.85154	80.84784	120.8458
.85	7.113172	9.095143	11.08398	21.06080	31.05282	51.04633	81.04264	121.0406
.90	7.356958	9.339466	11.32857	21.30578	31.29787	51.29143	81.28775	121.2857
.95	7.718391	9.701640	11.69111	21.66887	31.66108	51.65469	81.65104	121.6490
.97	7.953181	9.936878	11.92658	21.90468	31.89696	51.89061	81.88697	121.8849
.99	8.396685	10.38117	12.37128	22.34999	32.34240	52.33612	82.33251	122.3305
.995	8.645082	10.62997	12.62029	22.59934	32.59182	52.58558	82.58198	122.5800
.999	9.157380	11.14304	13.13378	23.11348	33.10609	53.09994	83.09636	123.0943
.9999	9.783802	11.77031	13.76151	23.74194	33.73473	53.72866	83.72513	123.7231
.99999	10.32779	12.31496	14.30652	24.28755	34.28047	54.27449	84.27098	124.2690
.999999	10.81475	12.80245	14.79431	24.77585	34.76889	54.76298	84.75950	124.7575

following ranges:

$$P = 0.01(.01)0.99,$$
$$D = 0(.1)5(.2)10(2)20(5)120,$$

and

$$P = .99(.0005).9990(.0001).9999(.00001).99999(.000001).999999,$$
$$D = 0, .05, .10, .25, .75, 1, 1.5, 2, 3, 4, 5, 6, 8, 10, 20, 30, 50, 80, 120.$$

This table required the calculation of over 45,000 $P(R, D)$ functions to an accuracy of seven or more decimal digits. The tabulated values of R, determined by a Newton-Raphson process, are correct to within one unit in the last digit position given. The method by which this conclusion was verified is given in Appendix C of [4]. A condensed version of the complete table is given below. The complete table as well as a similar one for K as a function of V and c are available by direct request to the authors.

It can be proved that $R(P, D)$ as a function of P approximates a univariate normal distribution to any desired accuracy for sufficiently large fixed values of D and $|R - D|/(R + D) \ll 1$. The relation between R and P in this case is given by

$$P(R, D) \approx \frac{1}{2}\left[1 + \operatorname{Erf}\left(\frac{R - \mu_R}{\sqrt{2}}\right)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{R-\mu_R} e^{-t^2/2}\, dt, \tag{39}$$

where $\mu_R \equiv R(0.50, D) \approx D + 1/(2D)$. (A slightly different formulation of the asymptotic behavior was given by Germond in [7]). This shows that the functional relationship is symmetric with respect to the point $R = \mu_R$, $P = 0.50$. This is evident from a study of Table 1. Also, if $20 \leqq D \leqq 25$, and if μ_R is computed from the approximation $D + 1/(2D)$ (which for these values of D is accurate to 10^{-5} or better), and if values of R as a function of P are then computed from equation (39) by inverse interpolation in an error function or univariate probability integral table, the results are, in general, correct within 10^{-3}, or one unit in the fifth significant figure of R. Further, the accuracy improves rapidly as D increases. This means that an efficient inverse table such as Table 1 need extend only from $P = 0$ to $P = 0.50$ if D is large. Each value of R for $P > 0.50$ is then found with only one subtraction and one addition by using the symmetry property stated above.

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