



It follows upon evaluating these derivatives at  $\phi = 0$  and noting that  $\phi = k^2$ ,

$$(3) \quad \begin{aligned} sn(u, k^2) = & \sin u - \frac{\cos u}{4} \left( u - \frac{\sin 2u}{2} \right) k^2 \\ & - \frac{\cos u}{4} \left[ \frac{3}{4} \left( \frac{3u}{2} - \sin 2u + \frac{\sin 4u}{8} \right) - \sin^2 u \left( u - \frac{\sin 2u}{2} \right) \right. \\ & \left. + \frac{\tan u}{4} \left( u - \frac{\sin 2u}{2} \right)^2 \right] \frac{k^4}{2!} - \dots + \left\{ \frac{\partial^m sn}{\partial \phi^m} \right\}_{\phi=0} \cdot \frac{k^{2m}}{m!} - \dots \end{aligned}$$

Expansions for other Jacobian functions can be generated from the relations between these functions and  $sn$ . The results are given explicitly up to  $k^4$  in the following:

$$(4) \quad \begin{aligned} cn(u, k^2) = & \cos u + \frac{\sin u}{4} \left( u - \frac{\sin 2u}{2} \right) k^2 - \left\{ \frac{\cos u}{16} \left( u - \frac{\sin 2u}{2} \right)^2 \right. \\ & \left. + \frac{\sin u}{4} \left[ \sin^2 u \left( u - \frac{\sin 2u}{2} \right) - \frac{3}{4} \left( \frac{3}{2} u - \sin 2u + \frac{\sin 4u}{8} \right) \right] \right\} \frac{k^4}{2!} + \dots \end{aligned}$$

$$(5) \quad dn(u, k^2) = 1 - \frac{1}{2} (\sin^2 u) k^2 + \left( \frac{u \sin 2u}{4} - \frac{\sin^2 2u}{8} - \frac{1}{4} \sin^4 u \right) \frac{k^4}{2!} + \dots$$

$$(6) \quad \begin{aligned} ns(u, k^2) = & \csc u + \csc^2 u \left( \frac{\cos u}{4} \right) \left( u - \frac{\sin 2u}{2} \right) k^2 + \left[ -\frac{\cos u}{4} \left( u - \frac{\sin 2u}{2} \right) \right. \\ & \left. - \frac{3}{16} \cos u \csc^2 u \left( \frac{3}{2} u - \sin 2u + \frac{\sin 4u}{8} \right) \right. \\ & \left. + \frac{1}{16} \csc u (\csc^2 u + \cot^2 u) \left( u - \frac{\sin 2u}{2} \right)^2 \right] \frac{k^4}{2!} + \dots \end{aligned}$$

$$(7) \quad \begin{aligned} cs(u, k^2) = & \cot u + \frac{\csc^2 u}{4} \left( u - \frac{\sin 2u}{2} \right) k^2 + \left[ -\frac{1}{8} \left( u - \frac{\sin 2u}{2} \right)^2 \cot u \csc^2 u \right. \\ & \left. + \frac{3}{16} \csc^2 u \left( \frac{3}{2} u - \sin 2u + \frac{\sin 4u}{8} \right) - \frac{1}{4} \left( u - \frac{\sin 2u}{2} \right) \right] \frac{k^4}{2!} + \dots \end{aligned}$$

$$(8) \quad \begin{aligned} ds(u, k^2) = & \csc u + \left[ \frac{1}{4} \cot u \csc u \left( u - \frac{\sin 2u}{2} \right) - \frac{1}{2} \sin u \right] k^2 \\ & + \frac{1}{4} \left[ -\sin^3 u + \left( u - \frac{\sin 2u}{2} \right)^2 \left( \frac{\csc u}{4} \right) (\csc^2 u + \cot^2 u) \right. \\ & \left. + \frac{3}{4} \cot u \csc u \left( \frac{3}{2} u - \sin 2u + \frac{\sin 4u}{8} \right) \right] \frac{k^4}{2!} + \dots \end{aligned}$$

$$(9) \quad \begin{aligned} sc(u, k^2) = & \tan u - \frac{1}{4} \sec^2 u \left( u - \frac{\sin 2u}{2} \right) k^2 \\ & - \frac{1}{4} \left[ \frac{3}{8} (1 + \sec^2 u) \left( \frac{3}{2} u - \sin 2u + \frac{\sin 4u}{8} \right) \right. \\ & \left. - \left( u - \frac{\sin 2u}{2} \right) \tan^2 u - \frac{1}{2} \left( u - \frac{\sin 2u}{2} \right)^2 \tan u \sec^2 u \right] \frac{k^4}{2!} - \dots \end{aligned}$$

$$\begin{aligned}
 dc(u, k^2) &= \sec u - \frac{1}{4} \tan u \left[ 2 \sin u + \sec u \left( u - \frac{\sin 2u}{2} \right) \right] k^2 \\
 &+ \left[ \frac{\sec u}{4} \left( u \sin 2u - \frac{\sin^2 2u}{2} - \sin^4 u \right) + \frac{1}{2} \tan^2 u \sin u \left( u - \frac{\sin 2u}{2} \right) \right. \\
 (10) \quad &- \frac{3}{16} \sin u \sec^2 u \left( \frac{3}{2} u - \sin 2u + \frac{\sin 4u}{8} \right) \\
 &\quad \left. - \frac{1}{16} \sec u \left( u - \frac{\sin 2u}{2} \right)^2 (\tan^2 u + \sec^2 u) \right] \frac{k^4}{2!} + \dots
 \end{aligned}$$

$$\begin{aligned}
 nc(u, k^2) &= \sec u - \frac{\tan u \sec u}{4} \left( u - \frac{\sin 2u}{2} \right) k^2 \\
 (11) \quad &+ \left[ \left( \frac{1}{16} \sec u + \frac{1}{8} \tan^2 u \sec u \right) \left( u - \frac{\sin 2u}{2} \right)^2 + \frac{1}{4} \tan^2 u \sin u \left( u - \frac{\sin 2u}{2} \right) \right. \\
 &\quad \left. + \frac{3}{4} \sec^2 u \left( \frac{3}{2} u - \sin 2u + \frac{\sin 4u}{8} \right) \right] \frac{k^4}{2!} + \dots
 \end{aligned}$$

$$(12) \quad nd(u, k^2) = 1 + \frac{\sin^2 u}{2} k^2 - \frac{1}{4} \left[ u \sin 2u - \frac{\sin^2 2u}{2} - 3 \sin^4 u \right] \frac{k^4}{2!} + \dots$$

TABLE 1  
Computed Jacobian Elliptic Functions

u	Sin <sup>-1</sup> k	sn					cn				
		k-Series		u-Series		Exact Value	k-Series		u-Series		Exact Value
		Com- puted Value	% Er- ror	Com- puted Value	% Er- ror		Com- puted Value	% Er- ror	Com- puted Value	% Er- ror	
.1745	5°	.17361	.02	.17361	.02	.17365	.98481	0	.98481	0	.98481
.1754	80°	.17365	0	.17365	0	.17365	.98481	0	.98654	.18	.98481
.7016	15°	.64277	0	.64294	.02	.64279	.76606	0	.76668	.08	.76604
.9401	50°	.76597	.01	.77886	1.67	.76604	.64282	0	.66704	3.77	.64279
1.1424	45°	.86587	.02	.90344	4.32	.86603	.50024	.05	.56037	12.07	.50000
1.3372	40°	.93964	.01	1.02184	8.74	.93969	.34250	.14	.45935	34.3	.34202
1.5738	5°	.99999	0	1.000821	.82	1.0000	0	0	.02496		0

u	Sin <sup>-1</sup> k	dn				Exact Value
		k-Series		u-Series		
		Com- puted Value	% Er- ror	Com- puted Value	% Er- ror	
.1745	5°	.99986	0	.99988	0	.99988
.1754	80°	.98525	0	.98698	.17	.98527
.7016	15°	.98606	0	.98626	.02	.98606
.9401	50°	.80898	.09	.82828	2.29	.80972
1.1424	45°	.79147	.12	.83341	5.42	.79056
1.3372	40°	.80027	.41	.87352	9.61	.79697
1.5738	5°	.99620	0	.99837	.22	.99619

$$\begin{aligned}
 (13) \quad cd(u, k^2) = & \cos u + \frac{\sin u}{4} \left( u + \frac{\sin 2u}{2} \right) k^2 + \left[ -\frac{\cos u}{16} \left( u - \frac{\sin 2u}{2} \right)^2 \right. \\
 & + \frac{3}{16} \sin u \left( \frac{3}{2} u - \sin 2u + \frac{\sin 4u}{8} \right) \\
 & \left. - \frac{\cos u}{4} \left( u \sin 2u - \frac{\sin^2 2u}{2} - 3 \sin^4 u \right) \right] \frac{k^4}{2!} + \dots .
 \end{aligned}$$

$$\begin{aligned}
 (14) \quad sd(u, k^2) = & \sin u - \left[ \frac{\cos u}{4} \left( u - \frac{\sin 2u}{2} \right) - \frac{\sin^3 u}{2} \right] k^2 \\
 & - \left\{ \frac{\cos u}{4} \left[ \frac{3}{4} \left( \frac{3u}{2} - \sin 2u + \frac{\sin 4u}{8} \right) + \frac{\tan u}{4} \left( u - \frac{\sin 2u}{2} \right)^2 \right] \right. \\
 & \left. + \frac{\sin u}{4} \left( u \sin 2u - \frac{\sin^2 2u}{2} - 3 \sin^4 u \right) \right\} \frac{k^4}{2!} - \dots .
 \end{aligned}$$

Representative results computed for basic Jacobian functions using the  $k$ -series as well as the  $u$ -series are given in Table 1. The computations were carried to the first three terms for both types of series. The various  $k$ - $u$  values, representing typical combinations in different ranges, were picked from the table of elliptic integrals [2], so that the "exact" values of the functions are known and that the relative accuracies of the two types of series may be compared. The percentage errors are computed to the nearest .01. It is seen that the  $k$ -series are nearly always more accurate than the corresponding  $u$ -series, particularly at larger  $u$ -values.

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1. H. BATEMAN, *Higher Transcendental Functions*, v. II., McGraw-Hill Book Company, New York, 1953, p. 344.
2. E. JAHNKE & F. E. EMDE, *Table of Functions*, Fourth Edition, Dover, New York, 1945.