37 [L].—V. N. FADDEYEVA & N. M. TERENT'EV, *Tables of Values of the Function* $w(z) = e^{-z^2}\left(1 + \frac{2i}{\sqrt{\pi}} \int_0^z e^{t^2}\, dt\right)$, *for Complex Argument*, translated by D. G. FRY, Pergamon Press, New York, 1961, 280 p., 26 cm. Price $15.00.

This is a translation from the Russian edition, which appeared in 1954. The present volume is essentially a tabulation of the error function in the complex plane, and, with $z = x + iy$, $w(z) = u(x, y) + iv(x, y)$, gives 6D values of u and v. In Table 1 the range is $0 \leqq x$ 3, $0 \leqq y \leqq 3$, with spacing in each variable of 0.02. In Table 2 the range is $3 \leqq x \leqq 5$, $0 \leqq y \leqq 3$, and $0 \leqq x \leqq 5$, $3 \leqq y \leqq 5$, with spacing in each variable of 0.1. A formula is presented so that the table is everywhere interpolable to an accuracy within two units in the last place. For interpolation about z_0, the formula uses the data at $z_0 \pm h$ and $z_0 \pm ih$, where h is the spacing. Let

$$2\bar{\Delta}_x f(x, y) = f(x + h, y) - f(x - h, y)$$
$$2\epsilon = (\bar{\Delta}_x u - \bar{\Delta}_y v) + i(\bar{\Delta}_y u + \bar{\Delta}_x v)$$
$$\Delta_x^2 f(x, y) = f(x + h, y) - 2f(x, h) + f(x - h, y)$$
$$\tilde{\Delta}_x w = \bar{\Delta}_x w - \epsilon.$$

The interpolation formula reads

$$w(z) \sim w(z_0) + h\tilde{\Delta}_x w + \tfrac{1}{2}h^2\Delta_x^2 w,$$

and values of $\tilde{\Delta}_x u$, $\tilde{\Delta}_x v$, $\Delta_x^2 u$ and $\Delta_x^2 v$ are provided.

The foreword, written by V. A. Fok, enunciates applications of the tables to physical problems. Some properties of $w(z)$ are also given. The authors' introduction gives various representations of $w(z)$, including power series, asymptotic expansions, and continued fractions. The method of constructing and checking the table is discussed. To find values of $w(z)$ accurate to 6D outside the tabulated range, some approximations based on the continued fraction expansion are presented.

Other tables of the error function for complex argument have appeared since the original issue of the present tables. See, for example, *Math. Comp.*, v. 14, 1960, p. 83. In this table, as well as the one under review, z is in rectangular form. For tables of the error function with z in polar form, see *MTAC*, v. 7, 1953, p. 178 and *Math. Comp.*, v. 12, 1958, p. 304–305; v. 14, 1960, p. 84.

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38 [L].—I. E. KIREEVA & K. A. KARPOV, *Tablitsy Funktsiĭ Vebera*, (*Tables of Weber Functions*), Vol. 1, Vychislitel'nyi Tsentr, Akad. Nauk SSSR, Moscow, 1959, xxiv + 340 p., 27 cm. Price 37 rubles. [An English translation by Prasenjit Basu has been published in 1961 by Pergamon Press, New York. Price $20.00].

Weber's equation

$$y'' - \left(a + \frac{z^2}{4}\right) y = 0 \qquad (1)$$

is satisfied when $-a = p + \frac{1}{2}$ by Whittaker's function $D_p(z)$. If $y(a, z)$ is a solution, so also are $y(a, -z)$, $y(-a, iz)$, and $y(-a, -iz)$. The solutions for $a - 1, a, a + 1$ are connected by a linear recurrence relation.

If we replace a by $-i\alpha$, and z by $\zeta e^{i\pi/4}$, the equation becomes

$$y'' - (\alpha - \tfrac{1}{4}\zeta^2)y = 0, \tag{2}$$

which also has real solutions for real α, ζ. This has no recurrence relation connecting solutions for α, $\alpha - 1$, $\alpha + 1$, though there is a complex relation connecting those for α, $\alpha - i$, $\alpha + i$.

The equation (1) is usually taken as basic in the complex variable theory of these differential equations.

There are several special cases worth mentioning.

(i) If $p = a - \frac{1}{2}$ is an integer $\geqq 0$, $D_p(z) = e^{-z^2/4}h_n(z)$,

where
$$h_n(z) = (-1)^n e^{z^2/2} \frac{d^n}{dx^n} e^{-z^2/2}$$

is a Hermite polynomial.

(ii) If $p = -q - 1$ is an integer $\geqq 0$, a solution is $e^{z^2/4}h_n^*(z)$, in which $h_n^*(z) = (-i)^n h_n(iz)$, which is again a real polynominal.

(iii) The case of integral p also includes solutions which involve repeated integrals of $e^{\pm z^2/2}$.

(iv) When $a = -p - \frac{1}{2}$ is an integer, the solutions involve a finite series of Bessel functions of order $\frac{2k + 1}{4}$ (k an integer) and argument $\frac{z^2}{4}$. For instance, when $a = 0$, and z is real, two solutions are

$$\sqrt{\frac{z}{2}}\, K_{1/4}\left(\frac{z^2}{4}\right) \quad \text{and} \quad \sqrt{\frac{z}{2}}\left\{I_{1/4}\left(\frac{z^2}{4}\right) + I_{-1/4}\left(\frac{z^2}{4}\right)\right\}.$$

All these are relevant to equation (1), with real a and z. With equation (2) the only real case of interest is when $\alpha = 0$, and ζ is real, in which case

$$\sqrt{\zeta}\, J_{\pm 1/4}\left(\frac{\zeta^2}{4}\right)$$

are both solutions.

These cases are all considered in [1], where there are also tables for real solutions to equation (2) with $\pm\alpha = 0(1)10$.

The tables now under review break new ground. They are concerned with the case where $p = -a - \frac{1}{2}$ is real, while $z = x(1 + i)$, with x real. The reality of p means that a real recurrence relation remains in existence connecting solutions for $p - 1, p, p + 1$. The differential equation now becomes

$$\frac{d^2y}{dx^2} - \{(2p + 1)i - x^2)\}y = 0$$

and the solutions are no longer real, except when $2p + 1 = 0$.

The tables give $D_p\{x(1 + i)\} = u_p(x) + iv_p(x)$ for $\pm x = 0(.01)5$, $p = 0(.1)2$, and $\pm x = 5(.01)10$, $p = 0(.05)2$, generally to 6 decimals, except that there are only 5 decimals when $|x| \geqq 7.5$ and $p \geqq 1.8$ simultaneously.

Since p is real, the functions remain elementary when $p = 0, 1, 2$. In fact,

$$u_0(x) = \cos\frac{x^2}{2} \qquad v_0(z) = -\sin\frac{x^2}{2}$$

$$u_1(x) = x\left(\cos\frac{x^2}{2} + \sin\frac{x^2}{2}\right) \qquad v_1(z) = x\left(\cos\frac{x^2}{2} - \sin\frac{x^2}{2}\right)$$

$$u_2(x) = -\cos\frac{x^2}{2} + 2x^2\sin\frac{x^2}{2} \qquad v_2(x) = \sin\frac{x^2}{2} + 2x^2\cos\frac{x^2}{2}$$

are given in the Introduction, formula (6′).

When $p = -\frac{1}{2}$, the equation (3) becomes a real equation and has real solutions; it is then a case of (2) with $\zeta = \sqrt{2}x$. However, $D_{-1/2}\{x(1 + i)\} = u_{-1/2}(x) + iv_{-1/2}(x)$ is not real, and we find

$$\begin{aligned} u_{-1/2}(x) - iv_{-1/2}(x) &= 2^{-3/4}\sqrt{\pi x}J_{-1/4}\left(\frac{x^2}{2}\right) \\ &= \{2^{-1/4}\sqrt{\pi}/\Gamma(\tfrac{3}{4})\}\, U_1(x, 2) \\ &= 2^{-3/4}\{W(0, x\sqrt{2}) + W(0, -x\sqrt{2})\} \\ &= -2^{-5/4}\sqrt{\pi x}J_{1/4}\left(\frac{x^2}{2}\right) \\ &= -\{2^{-7/4}\sqrt{\pi}/\Gamma(\tfrac{5}{4})\}\, U_2(x, 2) \\ &= 2^{-5/4}\{W(0, x\sqrt{2}) - W(0, -x\sqrt{2})\} \end{aligned}$$

where $W(a, x)$ is the function tabulated in [1], while $U_1(x, 2)$ and $U_2(x, 2)$ are those tabulated by Smirnov [2, p. 121].

The case $p = -\frac{1}{2}$, or $a = 0$, is in many senses the "central" case; it gives the simplest Bessel function representation, and it is rather surprising that it is not tabulated in the work under review. True, it is obtainable from other published tables, but so also are the cases $p = \frac{1}{2}, \frac{3}{2}$ as we see below; yet these two values of p occur in the tables, although the cases $p = -\frac{3}{2}, -\frac{5}{2}$, of equal complexity and involving the same Bessel functions, do not.

For $p = \frac{1}{2}, \frac{3}{2}$, consider first the derivative $\dfrac{d}{dx} D_{-1/2}\{x(1 + i)\}$;

we have

$$\begin{aligned} u'_{-1/2}(x) - v'_{-1/2}(x) &= -2^{-3/4}\sqrt{\pi x^3}J_{3/4}\left(\frac{x^2}{2}\right) \\ &= \{2^{-1/4}\sqrt{\pi}/\Gamma(\tfrac{3}{4})\}\, U'(x, 2) \\ &= 2^{-1/4}\{W'(0, x\sqrt{2}) - W'(0, -x\sqrt{2})\} \\ &= -2^{-5/4}\sqrt{\pi x^3}J_{-3/4}\left(\frac{x^2}{2}\right) \\ &= -\{2^{-7/4}\sqrt{\pi}/\Gamma(\tfrac{5}{4})\}\, U_2'(x, 2) \\ &= 2^{-3/4}\{W'(0, x\sqrt{2}) + W'(0, -x\sqrt{2})\} \end{aligned}$$

where the prime indicates differentiation with respect to the argument λx of the function, concerned, whether this be $+1$, -1, $+\sqrt{2}$, or $-\sqrt{2}$; for example, $W'(0, -x\sqrt{2}) = -\sqrt{2}\frac{d}{dx}W(0, -x\sqrt{2})$.

Then

$$u_{1/2} = \frac{x}{2}(u_{-1/2} - v_{-1/2}) - \frac{1}{2}(u'_{-1/2} + v'_{-1/2})$$

$$v_{1/2} = \frac{x}{2}(u_{-1/2} + v_{-1/2}) + \frac{1}{2}(u'_{-1/2} - v'_{-1/2})$$

and

$$u_{3/2} = x(u_{1/2} - v_{1/2}) - \tfrac{1}{2}u_{-1/2}$$

$$v_{3/2} = x(u_{1/2} + v_{1/2}) - \tfrac{1}{2}v_{-1/2}.$$

None of these representations as Bessel functions is mentioned in the work reviewed.

The introduction (the reviewer has relied here on the translation by Presenjit Basu for the English edition) is an interesting one, describing properties of the function relevant to its original computation and numerical use, and including five pages of graphs and diagrams illustrating the behavior of the functions. Interpolation is considered with respect to both x and p, and the use of the recurrence relation for $p < 0$ or $p > 2$ is explained. There is no mention in the Introduction of special applications of the particular solutions given of equation (3), and none is known to the reviewer, although there are many for the solutions of equations (1) and (2), where parabolic cylinder coordinates are appropriate.

The Introduction contains also two auxiliary tables. The first (p. xx, xxi) gives $a_r = \pm p(p-1)(p-2) \cdots (p-2r+1)/2^{4r}r!$, and $b_r = \pm p(p+1) \cdots (p+2r-1)/2^{4r}r!$ for $r = 0(1)3$; these are coefficients in asymptotic expansions. The other table gives $\Gamma\left(\frac{1}{2} - \frac{p}{2}\right) \Big/ 2^{1+(p/2)}\sqrt{\pi}$ and $\Gamma\left(-\frac{p}{2}\right) \Big/ 2^{1+(p/2)}\sqrt{2\pi}$. In each case ten-figure values are given for $p = .05(.05)1.95$, $p = 1$ excluded.

The tables are arranged so that each opening gives $u_p(x)$, $v_p(x)$, $u_p(-x)$, $v_p(-x)$ for four values of p, at interval 0.1 or 0.05, and 51 values of x at interval 0.01, with extra lines (up to three) at the foot to assist in interpolation. The openings are grouped into subtables, with p running from 0 to 2 (with $p = 2$ standing alone on the last page, facing the next title), and a single block of values of x covering half a unit; there are thus 20 such subtables.

The photographic printing of typescript is reasonably good and clear, although exhibiting the familiar variation in blackness that is rather too common.

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1. J. C. P. MILLER, Editor, *Tables of Weber Parabolic Cylinder Functions*, Her Majesty's Stationery Office, London, 1955.

2. A. D. SMIRNOV, *Tablitsy Funktsiĭ Eiri i Spetsial'nykh Vyrozhdennykh Gipergeometricheskikh Funktsiĭ ...* (*Tables of Airy Functions and of Special Confluent Hypergeometric Functions ...*) Izdatel'stvo Akad. Nauk SSSR, Moscow, 1955. Edition in English, Pergamon Press, New York, 1960. See *MTAC*, v. 12, 1958, p. 84–86.