# Minimum Periods, Modulo $p$, of First-Order Bell Exponential Integers

By Jack Levine and R. E. Dalton

**1. Introduction.** The integers of the title, $B(n)$, can be defined by the generating function, given by Bell [1, 2],

$$e^{e^x-1} = \sum_{n=0}^{\infty} B(n) \frac{x^n}{n!}. \tag{1.1}$$

These numbers have been known for a long time and have a variety of interesting interpretations which include:

(a) $B(n)$ = the number of rhyming schemes in a stanza of $n$ lines (attributed to Sylvester by Becker [3],

(b) $B(n)$ = the number of pattern sequences for words of $n$ letters, as used in cryptology, Levine [4],

(c) $B(n)$ = number of ways $n$ unlike objects can be placed in 1, 2, 3, $\cdots$, or $n$ like boxes (allowing blank boxes), Whitworth [5, p. 88],

(d) $B(n)$ = number of ways a product of $n$ (distinct) primes may be factored, Jordan [6, p. 179], Williams [7].

Epstein [8] extended the definition of $B(n)$ to include all real and complex numbers $n$ by means of the representation

$$B(n) = \frac{1}{e} \sum_{t=0}^{\infty} \frac{t^n}{t!}. \tag{1.2}$$

He also gave several asymptotic formulas for $B(n)$ in addition to the numerical values of $B(n)$ for $n = 1, \cdots, 20$. This paper, as well as [2], contains numerous references dealing with these numbers.

For computational purposes, various defining relations are known, for example,

$$B(n) = \sum_{r=1}^{n} \sum_{k=0}^{r} \frac{(-1)^k}{r!} \binom{r}{k} (r-k)^n, \tag{1.3}$$

given by Bell [1], and Mendelsohn and Riordan [9]. This formula, (1.3), is equivalent to

$$B(n) = \sum_{r=1}^{n} S(n, r), \tag{1.4}$$

where $S(n, r)$ are Stirling numbers of the second kind, and which was obtained by Broggi [10] and Becker and Riordan [11]. Other references relative to (1.3) and (1.4) are found in Epstein [8].

$$B(n+1) = (B+1)^n, \tag{1.5}$$

where on the right, $B^m$ is to be replaced by $B(m)$ after expansion, was given by d'Ocogne [12]. (See also [1, 2, 11]).

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The difference formula,

$$B(n) = \Delta^n B(1), \tag{1.6}$$

Becker and Browne [3], was found to be the simplest for a digital computer, and was used in the computation of the $B(n)$ given in the present paper.

For a study of arithmetic properties of $B(n)$, the congruence of Touchard [13],

$$B(n+p) \equiv B(n) + B(n+1), \qquad \text{mod } p, \tag{1.7}$$

for $p$ a prime, is basic.

In addition, for our purposes, we mention the following congruence given by Hall [14], Touchard [13], and Williams [7],

$$B(n+p^m) \equiv B(n+1) + mB(n), \qquad \text{mod } p. \tag{1.8}$$

It is known that the (minimum) period of the sequence (reduced mod $p$)

$$B(0), B(1), B(2), \cdots, B(n), \cdots \tag{1.9}$$

is a divisor of

$$N_p = \frac{p^p - 1}{p - 1}; \tag{1.10}$$

and Williams [7] has shown this minimum period is precisely $N_p$ for $p = 2, 3, 5$.

In this paper we extend these results to primes $p > 5$. The results obtained are stated in the theorem below.

THEOREM. *The minimum period, mod p, of the sequence* $B(0), B(1), \cdots, B(n)$, *of first-order Bell exponential integers is* $N_p$ *for* $p = 7, 11, 13,$ *and* 17. *For the remaining primes* $p < 50$, $p = 19, 23, 29, 31, 37, 41, 43, 47,$ *no known proper divisor,* $N$, *of* $N_p$, *with* $N \leqq 10^{40}$ *can be a period.*

In the course of the computations connected with this theorem the results of Cunningham [15] on factoring $N_p$ have been extended to include several new factors for certain $p$. These are exhibited in Table 3.

In addition, the values of $B(n)$, $n \leqq 74$, have been computed, and are given in Table 1. This extends results of Gupta [16] for $n \leqq 50$. Also, the values of $B(n)$, mod $p$, ($n \leqq p$, $p < 50$) are given in Table 2. Such values are needed in testing for periods.

**2. Computation of $B(n)$.** The symbolic binomial expansion (1.5), though useful in the computation of the first several $B(n)$, becomes bulky and time-consuming as $n$ increases, since each successive $B(n)$ computed by this iterative scheme requires $n - 2$ multiplications and $n$ additions involving larger numbers at each iteration. Formula (1.6), together with the initial values $B(0) = 1$, $B(1) = 1$, by contrast, requires but $n - 1$ additions for each new $B(n)$. (See Becker and Browne [3]). Such a difference formula as (1.6) is ideally suited for a digital computer, since it substitutes fixed-point addition for multiplication in which accuracy to the unit's digit must always be maintained. The only limitation which presented itself was the increasing size of the integers and differences involved. Using an octuple-precision addition subroutine, the numbers were generated on the difference table until a $B(n)$ or a difference exceeded 80 digits, the capacity of a standard IBM card. This

TABLE 1

*The Exponential Integers* $B(n)$, $0 \leqq n \leqq 74$

| | | | $n$ |
|---|---|---|---|
| | | 1 | 0 |
| | | 1 | 1 |
| | | 2 | 2 |
| | | 5 | 3 |
| | | 15 | 4 |
| | | 52 | 5 |
| | | 203 | 6 |
| | | 877 | 7 |
| | | 4140 | 8 |
| | | 21147 | 9 |
| | | 115975 | 10 |
| | | 678570 | 11 |
| | | 4213597 | 12 |
| | | 27644437 | 13 |
| | | 190899322 | 14 |
| | | 1382958545 | 15 |
| | 1 | 0480142147 | 16 |
| | 8 | 2864869804 | 17 |
| | 68 | 2076806159 | 18 |
| | 583 | 2742205057 | 19 |
| | 5172 | 4158235372 | 20 |
| | 47486 | 9816156751 | 21 |
| | 450671 | 5738447323 | 22 |
| | 4415200 | 5855084346 | 23 |
| | 44595886 | 9294805289 | 24 |
| | 463859033 | 2229999353 | 25 |
| | 4963124652 | 3618756274 | 26 |
| 5 | 4571704793 | 6059989389 | 27 |
| 61 | 6053940459 | 9934652455 | 28 |
| 713 | 3980193886 | 0275191172 | 29 |
| 8467 | 4901451180 | 9332450147 | 30 |
| 102933 | 5894622637 | 6485095653 | 31 |
| 1280646 | 7004990871 | 3818925644 | 32 |
| 16295958 | 9284600760 | 6764728147 | 33 |
| 211950393 | 8864036046 | 2388656799 | 34 |
| 2816002030 | 1956026656 | 3340426570 | 35 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | 3 | 8197147298 | 9481833997 | 5525681317 | 36 |
| | | | | 52 | 8683662085 | 5044790194 | 5575624941 | 37 |
| | | | | 746 | 2898920956 | 2533052309 | 9540639146 | 38 |
| | | | | 10738 | 8233307746 | 9283276885 | 7986425209 | 39 |
| | | | | 157450 | 5883912049 | 3128932434 | 4702531067 | 40 |
| | | | | 2351152 | 5077406176 | 2820069407 | 7243788988 | 41 |
| | | | | 35742549 | 1988726172 | 9135350865 | 6626642567 | 42 |
| | | | | 552950118 | 7971654843 | 2171469328 | 0737767385 | 43 |
| | | | | 8701963427 | 3870550890 | 2360053185 | 5797148876 | 44 |
| | | | 13 | 9258505266 | 2636696023 | 4705399365 | 4079693415 | 45 |
| | | | 226 | 5418219334 | 4940029284 | 8444470539 | 2276158355 | 46 |
| | | | 3745 | 0059502461 | 5111965053 | 4209643151 | 0120174682 | 47 |
| | | | 62891 | 9796303118 | 4154202104 | 5407184953 | 7746015761 | 48 |
| | | | 1072613 | 7154573358 | 4003422155 | 1859000263 | 3917247281 | 49 |
| | | | 18572426 | 8771078270 | 4382577671 | 8190891749 | 9221852770 | 50 |
| | | | 326398387 | 0004111524 | 8569518301 | 9158252441 | 9255819477 | 51 |
| | | | 5820533802 | 4195872785 | 4646270632 | 1859914950 | 3972126463 | 52 |
| | | 10 | 5292851801 | 4714166107 | 7812980215 | 8353492840 | 2714242132 | 53 |
| | | 193 | 1728758914 | 5618265728 | 9500692855 | 0325734983 | 2850302011 | 54 |
| | | 3593 | 3408596862 | 2831041960 | 1885980436 | 6106538872 | 6959079837 | 55 |
| | | 67756 | 8532064582 | 4322581483 | 0683714197 | 4597905321 | 6268760300 | 56 |
| | | 1294826 | 6194750696 | 4462616580 | 6338060009 | 1749160260 | 9372517195 | 57 |
| | | 25071363 | 5898429611 | 4560786627 | 4375749422 | 5301562344 | 5622326263 | 58 |
| | | 491767433 | 3630962165 | 9000944152 | 6248968535 | 9101824891 | 9168867818 | 59 |
| | | 9769393074 | 6700755298 | 6994066961 | 6754555502 | 4634775747 | 4482558637 | 60 |
| | 19 | 6523644715 | 4794148211 | 4228389322 | 7899633456 | 7346067337 | 0562378245 | 61 |
| | 400 | 2373048214 | 5478623052 | 2819234667 | 5449355269 | 6306024008 | 2269259738 | 62 |
| | 8250 | 7717004056 | 2488991245 | 6724304738 | 0284501901 | 3433711094 | 3817172961 | 63 |
| | 172134 | 1433573588 | 5093436996 | 3665272571 | 1255575751 | 8404975804 | 5339873395 | 64 |
| | 3633778 | 7854578993 | 2241525768 | 2767737441 | 4100369945 | 6043598236 | 5219287372 | 65 |
| | 77605907 | 2388436694 | 8215593085 | 7960017792 | 7780598875 | 1927803800 | 0759795263 | 66 |
| | 1676501284 | 3015234533 | 6721288085 | 4005182365 | 7483175898 | 8866047702 | 1013719409 | 67 |
| 3 | 6628224206 | 6961354788 | 3464061802 | 8539032699 | 1748479319 | 0948067172 | 5803995436 | 68 |
| 80 | 9212768387 | 9478363368 | 4627770706 | 6239391942 | 3239986492 | 7377173674 | 4420003007 | 69 |
| 1807 | 5003898340 | 5112375567 | 8442449836 | 9141305841 | 2344680979 | 0822799303 | 5088029195 | 70 |
| 40813 | 0093410464 | 2742599456 | 0096213470 | 6689859323 | 6369225324 | 4336559472 | 6056131962 | 71 |
| 931452 | 8182092653 | 2882514514 | 8352734180 | 6516792394 | 6744967255 | 7893570602 | 9134658745 | 72 |
| 21483462 | 3568478894 | 4527656055 | 1192833336 | 7140719361 | 2910039971 | 6139004370 | 1285425833 | 73 |
| 500690802 | 4247925379 | 7070764709 | 5772222046 | 3116781409 | 6591601595 | 3698116129 | 8714301202 | 74 |

TABLE 2

$B(n)$ *mod* $p$, $0 \leqq n \leqq p$, $p \leqq 50$

| $n$ | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 | 43 | 47 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| 3 | | 2 | 0 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 4 | | | 0 | 1 | 4 | 2 | 15 | 15 | 15 | 15 | 15 | 15 | 15 | 15 | 15 |
| 5 | | | 2 | 3 | 8 | 0 | 1 | 14 | 6 | 23 | 21 | 15 | 11 | 9 | 5 |
| 6 | | | | 0 | 5 | 8 | 16 | 13 | 19 | 0 | 17 | 18 | 39 | 31 | 15 |
| 7 | | | | 2 | 8 | 6 | 10 | 3 | 3 | 7 | 9 | 26 | 16 | 17 | 31 |
| 8 | | | | | 4 | 6 | 9 | 17 | 0 | 22 | 17 | 33 | 40 | 12 | 4 |
| 9 | | | | | 5 | 9 | 16 | 0 | 10 | 6 | 5 | 20 | 32 | 34 | 44 |
| 10 | | | | | 2 | 2 | 1 | 18 | 9 | 4 | 4 | 17 | 27 | 4 | 26 |
| 11 | | | | | 2 | 9 | 15 | 4 | 1 | 28 | 11 | 27 | 20 | 30 | 31 |
| 12 | | | | | | 11 | 11 | 5 | 20 | 13 | 15 | 0 | 27 | 27 | 0 |
| 13 | | | | | | 2 | 6 | 7 | 1 | 13 | 1 | 35 | 23 | 38 | 24 |
| 14 | | | | | | | 15 | 14 | 12 | 7 | 20 | 5 | 1 | 5 | 33 |
| 15 | | | | | | | 11 | 16 | 9 | 20 | 30 | 36 | 9 | 27 | 42 |
| 16 | | | | | | | 14 | 15 | 5 | 28 | 16 | 2 | 4 | 42 | 38 |
| 17 | | | | | | | 2 | 1 | 6 | 17 | 8 | 29 | 19 | 19 | 12 |
| 18 | | | | | | | | 10 | 6 | 16 | 1 | 21 | 17 | 1 | 22 |
| 19 | | | | | | | | 2 | 9 | 20 | 21 | 28 | 16 | 20 | 44 |
| 20 | | | | | | | | | 4 | 20 | 3 | 23 | 3 | 26 | 43 |
| 21 | | | | | | | | | 16 | 15 | 25 | 32 | 22 | 27 | 5 |
| 22 | | | | | | | | | 22 | 5 | 26 | 32 | 33 | 39 | 25 |
| 23 | | | | | | | | | 2 | 25 | 19 | 6 | 3 | 36 | 29 |
| 24 | | | | | | | | | | 7 | 16 | 34 | 23 | 42 | 3 |
| 25 | | | | | | | | | | 24 | 2 | 0 | 3 | 27 | 20 |
| 26 | | | | | | | | | | 11 | 15 | 26 | 38 | 41 | 10 |
| 27 | | | | | | | | | | 21 | 16 | 19 | 13 | 42 | 0 |
| 28 | | | | | | | | | | 18 | 17 | 12 | 13 | 27 | 23 |
| 29 | | | | | | | | | | 2 | 12 | 21 | 35 | 1 | 30 |
| 30 | | | | | | | | | | | 3 | 10 | 23 | 11 | 22 |
| 31 | | | | | | | | | | | 2 | 1 | 4 | 35 | 44 |
| 32 | | | | | | | | | | | | 35 | 2 | 21 | 18 |
| 33 | | | | | | | | | | | | 35 | 37 | 3 | 35 |
| 34 | | | | | | | | | | | | 26 | 34 | 28 | 46 |
| 35 | | | | | | | | | | | | 17 | 14 | 33 | 11 |
| 36 | | | | | | | | | | | | 6 | 35 | 32 | 37 |
| 37 | | | | | | | | | | | | 2 | 40 | 28 | 45 |
| 38 | | | | | | | | | | | | | 31 | 3 | 25 |
| 39 | | | | | | | | | | | | | 6 | 7 | 16 |
| 40 | | | | | | | | | | | | | 5 | 12 | 17 |
| 41 | | | | | | | | | | | | | 2 | 41 | 5 |
| 42 | | | | | | | | | | | | | | 17 | 13 |
| 43 | | | | | | | | | | | | | | 2 | 9 |
| 44 | | | | | | | | | | | | | | | 23 |
| 45 | | | | | | | | | | | | | | | 37 |
| 46 | | | | | | | | | | | | | | | 19 |
| 47 | | | | | | | | | | | | | | | 2 |

condition occurred during the computation of $B(75)$. The program, which used the SOAP I assembly program, was used on an IBM 650 to compute the 75 numbers in 73 minutes. A check was made with Gupta's highest value, $B(50)$, and the numbers were found to be identical.

**3. Factorization of $N_p$, ($p < 50$).** From a result of Fontene [17], it follows that all factors of $N_p$ are of the form $2kp + 1$, when $p$ is an odd prime. Using this information, a program was developed for the Univac 1105 in the USE compiler language. This simply involved successive division of $N_p$ by divisors of the form

$$P_k = 2pk + 1, \qquad k = 1, 2, 3, \cdots$$

until a zero remainder was reached. Since the routine was single-precision for the divisors, the $P_k$'s were limited in magnitude to one accumulator length on the Univac 1105, or to values $P_k < 2^{35}$.

Table 3 gives the $N_p$ and the factors thus obtained.

The following is a summary of new prime factors and other information not contained in Cunningham [15, p. 72].

Case $p = 17$. $N_{17}$ is completely factored into the three prime factors 10949, 1749233, 2699538733.

Case $p = 19$. No factors of $N_{19}$ have been found, but $N_{19}$ contains no factor <17,005,305

TABLE 3

*$N_p$'s and Prime Factors (Indicated by*)*

| $p$ | $N_p = \frac{p^p - 1}{p - 1}$ |
|---|---|
| 5 | $N_5$ = 781 = 11*·71* |
| 7 | $N_7$ = 1 37257 = 29*·4733* |
| 11 | $N_{11}$ = 2 85311 67061 = 15797*·1806113* |
| 13 | $N_{13}$ = 2523 95922 16021 = 53*·264031*·1803647* |
| 17 | $N_{17}$ = 51702 51636 78960 47761 = 10949*·1749233*·2699538733* |
| 19 | $N_{19}$ = 1099 12203 09223 96438 40221 No known prime factors |
| 23 | $N_{23}$ = 94911 21818 11268 72883 43196 77753 = 461*·1289*· 1597216194112486480522357 |
| 29 | $N_{29}$ = 9 17030 76898 61468 33772 08150 52610 77188 02981 = 59*·16763*·84449*·2428577*·14111459*·32037737880884399 |
| 31 | $N_{31}$ = 56897 24710 24107 86528 70214 34301 97715 85348 24481 No known prime factors |
| 37 | $N_{37}$ = 29 31981 93216 04953 92799 53613 49988 42485 03538 78009 36166 51181 = 149*·1999*·7993*. (quotient > 40 digits) |
| 41 | $N_{41}$ = 33271 94076 58177 99967 83498 10240 83656 39964 72332 54041 27485 81284 48841 = 83*. (quotient > 40 digits) |
| 43 | $N_{43}$ = 4129 46984 92929 20838 07232 88782 88579 08531 14434 61669 54570 31137 54094 99893 = 173*·6709*. (quotient > 40 digits) |
| 47 | $N_{47}$ = 84 30270 13796 61926 57970 97431 77268 05988 90944 54377 04795 47313 54904 95405 42692 40497 = 1693*· (quotient > 40 digits) |

Case $p = 23$. No new prime factors of $N_{23}$ have been found, but the third factor 1,597,216,194,112,486,480,522,357 contains no factor $<59,929,399$

Case $p = 29$. Four new prime factors of $N_{29}$ are 16763, 84449, 2428577, 14111459.

**4. Determination of minimum periods, mod $p$.** The knowledge of $B(1)$, $B(2)$, $\cdots$, $B(p)$, (or of any $p$ consecutive $B$'s) will determine the complete set of $B$'s, mod $p$. Hence, if $N$ be a factor of $N_p$, to test for a period of the sequence $\{B(n)\}$ mod $p$, it is sufficient to calculate $B(N + 1)$, $B(N + 2)$, $\cdots$, $B(N + p)$, mod $p$, and compare with $B(1)$, $B(2)$, $\cdots$, $B(p)$, mod $p$.

Furthermore, if $N_p$ can be expressed as a product of $r$ factors, it is not necessary to test all possible combinations of factors for periods, but merely the combinations of $r - 1$ factors. A positive result would indicate what further testings are necessary.

In case the complete factorization of $N_p$ into prime factors is unknown it may not be possible to find the minimum period.

The actual testing of the various factors for the period property was accomplished on an IBM 650. The program requires $N$, the factor to be tested; $p$, the particular prime; and $B(1)$, $B(2)$, $\cdots$, $B(p)$, mod $p$. These $B$'s were obtained from a modification of the program used to calculate Table 1 and are given in Table 2. The program used could test any factor less than $10^{40}$. It would, of course, be impractical to calculate every $B$ through $B(N + p)$, so a process of proceeding in jumps of powers of $p$ by means of (1.8) is used.

The factor $N$ being tested is first expressed to the base $p$,

$$N = a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p + a_0 . \tag{4.1}$$

The various steps are then (all calculations mod $p$):

(1) Calculate $B(p + 1)$ by (1.7).

(2) Calculate $B(a_n p^n + x)$, $x = 1, 2, \cdots, p + 1$, by the iterations

$$B(tp^n + y) = B((t - 1)p^n + y + 1) + nB((t - 1)p^n + y), \tag{4.2}$$

$$B(tp^n + p + 1) = B(tp^n + 1) + B(tp^n + 2), \tag{4.3}$$

where $t = 1, 2, \cdots, a_n$; $y = 1, 2, \cdots, p$. Equation (4.2) follows from (1.8), and (4.3) from (1.7).

(3) Calculate $B(a_n p^n + a_{n-1} p^{n-1} + x)$, $x = 1, 2, \cdots, p + 1$, by

$$B(up^{n-1} + z) = B((u - 1)p^{n-1} + z + 1) + (n - 1)B((u - 1)p^{n-1} + z), \tag{4.4}$$

$$B(up^{n-1} + p + 1) = B(up^{n-1} + 1) + B(up^{n-1} + 2), \tag{4.5}$$

where $u = 1, 2, \cdots, a_{n-1}$; $z = a_n p^n + 1, \cdots, a_n p^n + p$.

This procedure is continued until we reach

$$B(M + 1), B(M + 2), \cdots, B(M + p), \tag{4.6}$$

where

$$M = a_n p^n + a_{n-1} p^{n-1} + \cdots + a_1 p.$$

Since one member of (4.6) is $B(N)$, we start from that point and calculate

$$B(N+1), B(N+2), \cdots, B(N+p),$$

which are then compared with

$$B(1), B(2), \cdots, B(p),$$

for the period property. The results of these calculations have been given in the theorem of Section 1.

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