Polynomial Expansions of Bessel Functions and Some Associated Functions

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1. Introduction. In this paper we first determine representations for the Anger-Weber functions $\mathbf{J}_v(ax)$ and $\mathbf{E}_v(ax)$ in series of symmetric Jacobi polynomials. (These include Legendre and Chebyshev polynomials as special cases.) If v is an integer, these become expansions for the Bessel function of the first kind, since $\mathbf{J}_k(ax) = J_k(ax)$. In Section 3, corresponding representations are found for $(ax)^{-v}J_v(ax)$. Convenient error bounds are obtained for the above expansions. In the fourth section we determine the similar type expansions for the Bessel functions $Y_k(ax)$ and $K_k(ax)$. In Section 5, the coefficients of some of our expansions are tabulated for particularly important values of the various parameters.

2. Symmetric Jacobi Expansions of Anger-Weber Functions. A function $f(x)$ satisfying certain conditions (for these consult [1]) may be expanded in the series

$$f(x) = \sum_{n=0}^{\infty} C_n P_n^{(\alpha,\alpha)}(x), \quad -1 \leqq x \leqq 1, \quad \alpha > -1, \tag{2.1}$$

where $P_n^{(\alpha,\alpha)}(x)$ is called the symmetric Jacobi polynomial of degree n. For our present purposes we shall use a definition given in [2]:

$$2^n n! P_n^{(\alpha,\alpha)}(x) = (-)^n(1 - x^2)^{-\alpha}D^n[(1 - x^2)^{\alpha+n}]. \tag{2.2}$$

Also

$$C_n = h_n^{-1} \int_{-1}^{1} f(x)(1 - x^2)^{\alpha} P_n^{(\alpha,\alpha)}(x)\, dx, \tag{2.3}$$

$$h_n = \frac{2^{2\alpha}(n+1)_\alpha}{(n + \alpha + \frac{1}{2})(n + \alpha + 1)_\alpha}; \qquad (v)_\mu = \frac{\Gamma(v+\mu)}{\Gamma(v)}, \qquad (v)_0 = 1. \tag{2.4}$$

Using the representation (2.2) in (2.3) and noticing that all derivatives of $(1 - x^2)^{\alpha+n}$ up to and including the $(n-1)$st vanish at $x = \pm 1$, we integrate (2.3) n times by parts to get:

$$C_n = (2^n n!\, h_n)^{-1} \int_{-1}^{1} f^{(n)}(x)(1 - x^2)^{\alpha+n}\, dx. \tag{2.5}$$

Consider the integral definition of the Anger-Weber functions [2, v. 2, p. 35]

$$\mathbf{J}_v(ax) + i\mathbf{E}_v(ax) = \pi^{-1} \int_0^{\pi} e^{i[v\phi - ax \sin\phi]}\, d\phi = f(x). \tag{2.6}$$

When v is an integer, $\mathbf{J}_v(ax)$ coincides with the Bessel function of the first kind $J_v(ax)$ [2, v. 2, p. 4].

Now differentiate (2.6) n times under the integral sign, substitute the result in

Received February 27, 1962.

(2.5) and interchange the order of integration (which is, of course, permissible). The inner integral is known [3] and after evaluating it we have

$$C_n = (-i)^n (n+1)_\alpha (h_n \pi^{1/2})^{-1} \cdot \int_0^\pi e^{iv\phi} \left\{\frac{a \sin \phi}{2}\right\}^{-[\alpha+(1/2)]} J_{[n+\alpha+(1/2)]}(a \sin \phi)\, d\phi. \tag{2.7}$$

Use the power series expansion for the Bessel function in (2.7) and integrate term-by-term to get

$$C_n = (-i)^n \left[\cos \frac{v\pi}{2} + i \sin \frac{v\pi}{2}\right] \Lambda_n R_n(v, \alpha, a), \tag{2.8}$$

where

$$\Lambda_n = \frac{a^n n!}{\Gamma\left(\frac{n}{2} + \frac{v}{2} + 1\right) \Gamma\left(\frac{n}{2} - \frac{v}{2} + 1\right) (n + 2\alpha + 1)_n}, \tag{2.9}$$

and R_n is conveniently described in hypergeometric notation [2, v. 1, p. 182] as

$$R_n(v, \alpha, a) = {}_2F_3\left[\frac{n}{2} + \frac{1}{2}, \frac{n}{2} + 1; \alpha + n + \frac{3}{2}, \frac{n}{2} + \frac{v}{2} + 1, \frac{n}{2} - \frac{v}{2} + 1; -\frac{a^2}{4}\right] \tag{2.10}$$

Equating real and imaginary parts of (2.6) and (2.1), we get

$$\mathbf{J}_v(ax) = \sum_{n=0}^{\infty} A_n P_n^{(\alpha,\alpha)}(x), \quad -1 \leqq x \leqq 1, \tag{2.11}$$

$$\mathbf{E}_v(ax) = \sum_{n=0}^{\infty} B_n P_n^{(\alpha,\alpha)}(x), \quad -1 \leqq x \leqq 1, \tag{2.12}$$

where

$$A_n = \Lambda_n R_n(v, \alpha, a) \phi_n(v), \tag{2.13}$$

$$B_n = \Lambda_n R_n(v, \alpha, a) \psi_n(v), \tag{2.14}$$

and

$$\phi_n(v) = \begin{cases} (-)^{n/2} \cos \frac{v\pi}{2}, & n \text{ even}, \\ (-)^{(n-1)/2} \sin \frac{v\pi}{2}, & n \text{ odd}; \end{cases} \tag{2.15}$$

$$\psi_n(v) = \begin{cases} (-)^{n/2} \sin \frac{v\pi}{2}, & n \text{ even}, \\ (-)^{(n+1)/2} \cos \frac{v\pi}{2}, & n \text{ odd}. \end{cases} \tag{2.16}$$

Equations (2.11) and (2.12) and the expansions in Section 3 may also be derived from results in [4]. The present derivation is more satisfactory because it establishes a foundation for the work in Section 4.

When $\alpha = -\frac{1}{2}$,

$$P_n^{[-(1/2),-(1/2)]}(x) = (\tfrac{1}{2})_n (n!)^{-1} T_n(x), \quad n = 1, 2 \cdots, \tag{2.17}$$

where $T_n(x)$ is the Chebyshev polynomial of the first kind of degree n. Also for this value of α, R_n simplifies to the product of two Bessel functions [2, v. 2, p. 11]. With $\alpha = -\frac{1}{2}$, then (2.11)–(2.14) become

$$\mathbf{J}_v(ax) = \sum_{n=0}^{\infty} C_n T_n(x), \quad -1 \leqq x \leqq 1, \tag{2.18}$$

$$\mathbf{E}_v(ax) = \sum_{n=0}^{\infty} D_n T_n(x), \quad -1 \leqq x \leqq 1, \tag{2.19}$$

where

$$C_n = \epsilon_n J_{(n+v)/2}\left(\frac{a}{2}\right) J_{(n-v)/2}\left(\frac{a}{2}\right) \phi_n(v), \tag{2.20}$$

$$D_n = \epsilon_n J_{(n+v)/2}\left(\frac{a}{2}\right) J_{(n-v)/2}\left(\frac{a}{2}\right) \psi_n(v), \tag{2.21}$$

and $\epsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n > 0. \end{cases}$

For integral v we have the expansions

$$J_{2k}(ax) = \sum_{n=0}^{\infty} \epsilon_n J_{k+n}\left(\frac{a}{2}\right) J_{k-n}\left(\frac{a}{2}\right) T_{2n}(x), \quad -1 \leqq x \leqq 1, \tag{2.22}$$

$$J_{2k+1}(ax) = 2\sum_{n=0}^{\infty} J_{k+n+1}\left(\frac{a}{2}\right) J_{k-n}\left(\frac{a}{2}\right) T_{2n+1}(x), \quad -1 \leqq x \leqq 1, \tag{2.23}$$

and $k = 0, 1, 2 \cdots$. Equation (2.22) is known [2, v. 2, p. 100].

Since

$$J_v(iz) = e^{(v\pi i)/2} I_v(z), \tag{2.24}$$

where $I_v(z)$ is the modified Bessel function of the first kind [2, v. 2, p. 5], we may replace a by ia in (2.22) and (2.23) to get expansions for $I_{2k}(ax)$ and $I_{2k+1}(ax)$.

It is important to note that, although the above expansions are valid only for x real and $|x| \leqq 1$, (2.6) is entire in a and v, and hence a may be chosen arbitrarily to yield expansions valid anywhere in the finite complex plane.

The expansions (2.11), (2.12), (2.18), (2.19), (2.22), and (2.23) are quite rapidly convergent, particularly in the Chebyshev cases [5]; consequently the last four expansions are eminently suitable for use on digital computers.* Such series

* The Bessel functions required to compute the coefficients in our expansions can be systematically generated on electronic computers with the aid of techniques discussed in [6, 7, 8]. There are numerous tables available for hand calculations. The words "accuracy," "error," and "convergence" in this paper always refer to the properties of the expansion when truncated after a finite number of terms.

may be truncated and rearranged in powers of x. Clenshaw [9], though, by using the recursion formulas satisfied by the Chebyshev polynomials, has formulated a convenient nesting procedure which allows one to utilize such expansions directly. The scheme is as follows. Consider

$$f^{(1)}(x) = \sum_{n=0}^{N} A_n^{(1)} T_n^* \left(\frac{x}{a}\right), \quad 0 \leqq x \leqq a, \tag{2.25}$$

$$f^{(2)}(x) = \sum_{n=0}^{N} A_n^{(2)} T_{2n} \left(\frac{x}{a}\right), \quad -a \leqq x \leqq a, \tag{2.26}$$

$$f^{(3)}(x) = \sum_{n=0}^{N} A_n^{(3)} T_{2n+1} \left(\frac{x}{a}\right), \quad -a \leqq x \leqq a. \tag{2.27}$$

To evaluate the series (2.25), (2.26), or (2.27), respectively, we construct the following sequences:

$$b_n^{(1)} = \left[4\left(\frac{x}{a}\right) - 2\right] b_{n+1}^{(1)} - b_{n+2}^{(1)} + A_n^{(1)}, \tag{2.28}$$

$$b_n^{(2)} = \left[4\left(\frac{x}{a}\right)^2 - 2\right] b_{n+1}^{(2)} - b_{n+2}^{(2)} + A_n^{(2)}, \tag{2.29}$$

$$b_n^{(3)} = \left[4\left(\frac{x}{a}\right)^2 - 2\right] b_{n+1}^{(3)} - b_{n+2}^{(3)} + A_n^{(3)}, \tag{2.30}$$

for $n = N, N-1, N-2, \cdots, 3, 2, 1, 0$ with the initial values

$$b_{N+1}^{(1)} = b_{N+2}^{(1)} = b_{N+1}^{(2)} = b_{N+2}^{(2)} = b_{N+1}^{(3)} = b_{N+2}^{(3)} = 0.$$

$f^{(1)}(x), f^{(2)}(x)$, and $f^{(3)}(x)$ are then given by

$$f^{(1)}(x) = b_0^{(1)} + b_1^{(1)} \left[1 - 2\left(\frac{x}{a}\right)\right], \tag{2.31}$$

$$f^{(2)}(x) = b_0^{(2)} + b_1^{(2)} \left[1 - 2\left(\frac{x}{a}\right)^2\right], \tag{2.32}$$

$$f^{(3)}(x) = [b_0^{(3)} - b_1^{(3)}] \left(\frac{x}{a}\right). \tag{2.33}$$

The method is as direct as the ordinary nesting process used to evaluate polynomials.

We now derive error estimates for the expansions (2.11) and (2.12) for $-1 \leqq x \leqq 1$. Notice that

$$R_n(v, \alpha, a) = 1 + 0\left(\frac{1}{n}\right) \tag{2.34}$$

provided all other parameters are fixed, and consequently

$$| A_n | \leqq \frac{| a |^n \, n!}{\Gamma\left(\frac{n+v}{2} + 1\right) \Gamma\left(\frac{n-v}{2} + 1\right) (n + 2\alpha + 1)_n} \left| 1 + 0\left(\frac{1}{n}\right) \right|, \tag{2.35}$$

and likewise for B_n. Also [2, v. 2, p. 206]

$$\max_{-1 \leqq x \leqq 1} | P_n^{(\alpha,\alpha)}(x) | = \binom{n+\alpha}{n}, \quad \alpha \geqq -\tfrac{1}{2}. \tag{2.36}$$

Let ϵ_N denote the error incurred by taking just N terms of (2.11) or (2.12). Because of the rapidity of convergence of the expansions, as shown by (2.35), the $(N+1)$th term furnishes us with a convenient error estimate

$$| \epsilon_N | = \frac{| a |^N N^{\alpha+(1/2)} \pi^{1/2}}{2^{2N+2\alpha}\Gamma\left(\frac{N+v}{2}+1\right)\Gamma\left(\frac{N-v}{2}+1\right)\Gamma(\alpha+1)} \left| 1 + 0\left(\frac{1}{N}\right) \right|, \tag{2.37}$$

where $\alpha \geqq -\frac{1}{2}$, $N > v$, $-1 \leqq x \leqq 1$.

Among the values of α considered, it follows from (2.37) that the choice $\alpha = -\frac{1}{2}$, i.e., the Chebyshev case, yields the smallest error term for large N.

3. Expansions of Bessel Functions of the First Kind of Nonintegral Order. Results in the previous section gave symmetric Jacobi polynomial expansions for $J_v(ax)$ and $I_v(ax)$ for integral v. When v is nonintegral, these functions are no longer entire functions of x, and it is convenient to derive an expansion for the entire function

$$\Gamma(v+1)(ax/2)^{-v}J_v(ax) = {}_0F_1\left(v+1; -\frac{a^2x^2}{4}\right). \tag{3.1}$$

Corresponding expansions for $\Gamma(v+1)(ax/2)^{-v}I_v(ax)$ then follow, as before, from (2.24).

Let $f(x)$ in (2.5) be the right-hand side of (3.1). Then we have

$$J_v(ax) = (ax)^v \sum_{n=0}^{\infty} A_n P_{2n}^{(\alpha,\alpha)}(x), \quad -1 \leqq x \leqq 1, \tag{3.2}$$

where

$$\begin{aligned} A_n &= \frac{(-)^n(2a)^{2n}}{2^v\pi^{1/2}(2n+2\alpha+1)_{2n}(n+\frac{1}{2})_{v+(1/2)}} \\ &\quad \cdot {}_1F_2\left(n+\frac{1}{2};\ v+n+1,\ 2n+\alpha+\frac{3}{2};\ -\frac{a^2}{4}\right). \end{aligned} \tag{3.3}$$

These equations also follow from a result in [4]. Indeed, using a general expansion given there, an alternative formula for (3.3) can be stated. We have

$${}_1F_2\left[\rho; \sigma, \tau; -\frac{z^2}{4}\right] = \Gamma(\sigma)(z/2)^{1-\sigma} \sum_{k=0}^{\infty} \frac{(z/2)^k(\tau-\rho)_k}{k!\,(\tau)_k} J_{k+\sigma-1}(z), \tag{3.4}$$

$$\begin{aligned} A_n &= \frac{(-)^n 2^{(1/2)-\alpha-v}\Gamma(n+\frac{1}{2})(2n+\alpha+\frac{1}{2})(2n+\alpha+1)_\alpha}{a^{(1/2)+\alpha}} \\ &\quad \cdot \sum_{k=0}^{\infty} \frac{(a/2)^k(v+\frac{1}{2})_k}{k!\,\Gamma(v+n+k+1)} J_{2n+k+\alpha+(1/2)}(a). \end{aligned} \tag{3.5}$$

For the Chebyshev case of (3.2) $\alpha = -\frac{1}{2}$ and

$$J_v(ax) = (ax)^v \sum_{n=0}^{\infty} C_n T_{2n}(x), \quad -1 \leqq x \leqq 1, \tag{3.6}$$

where

$$C_n = \frac{\epsilon_n(-)^n(a/4)^{2n}}{2^v n!\,\Gamma(v+n+1)}\, {}_1F_2\left[n+\tfrac{1}{2};\quad v+n+1, 2n+1;\quad -\frac{a^2}{4}\right]. \tag{3.7}$$

Notice that when $v = -\frac{1}{2}$, (3.3) simplifies. Also, since

$$J_{-(1/2)}(ax) = \left(\frac{\pi ax}{2}\right)^{-(1/2)} \cos(ax), \tag{3.8}$$

we infer the expansion

$$\cos(ax) = \sum_{n=0}^{\infty} C_n P_{2n}^{(\alpha,\alpha)}(x), \quad -1 \leqq x \leqq 1, \tag{3.9}$$

where

$$C_n = \frac{(-)^n \pi^{1/2} 2^{-\alpha+(1/2)}(2n+\alpha+\frac{1}{2})(2n+\alpha+1)_\alpha}{a^{\alpha+(1/2)}} J_{2n+\alpha+(1/2)}(a), \tag{3.10}$$

a formula which can be derived in a number of different ways.

Using an analysis similar to that of Section 2, we may derive the estimate for the error incurred when just N terms of (3.2) are used.

$$| \epsilon_N | = \frac{|a|^{v+2N} |x|^v \pi^{1/2} N^{\alpha+(1/2)}}{2^{4N+\alpha+v-(1/2)} N!\,\Gamma(N+v+1)\Gamma(\alpha+1)} \left| 1 + 0\left(\frac{1}{N}\right)\right|, \tag{3.11}$$

$$-1 \leqq x \leqq 1, \quad \alpha \geqq -\tfrac{1}{2}, \quad N > v.$$

Concerning the optimum choice of α in (3.2), see the discussion surrounding (2.37).

4. Expansions of Bessel Functions of the Second Kind. The Bessel function and modified Bessel function of the second kind are denoted by $Y_v(z)$ and $K_v(z)$, respectively, and a treatment of them can be found in [2, v. 2, Ch. VII]. If v is nonintegral, then

$$Y_v(z) = [\sin(v\pi)]^{-1}\{J_v(z)\cos(v\pi) - J_{-v}(z)\}, \tag{4.1}$$

and

$$K_v(z) = \frac{\pi}{2}[\sin(v\pi)]^{-1}\{I_{-v}(z) - I_v(z)\}, \tag{4.2}$$

so for such values of v expansions for the functions follow directly from the results of Section 3.

If v is an integer, it can be shown that

$$Y_k(ax) = \frac{2}{\pi}\left[\gamma + \ln\left(\frac{ax}{2}\right)\right] J_k(ax) + N_{k-1}(ax) - \frac{1}{\pi} W_k(ax), \tag{4.3}$$

and

$$(4.4)\quad K_k(ax) = (-)^{k+1}\left[\gamma + \ln\left(\frac{ax}{2}\right)\right] I_k(ax) - \frac{\pi}{2}\, i^k N_{k-1}(iax) + \frac{i^k}{2} W_k(iax),$$

where

$$(4.5)\qquad N_{k-1}(ax) = \begin{cases} -\dfrac{1}{\pi}\displaystyle\sum_{m=0}^{k-1}\left(\frac{ax}{2}\right)^{2m-k}\frac{(k-m-1)!}{m!}, & k > 0 \\ 0, & k = 0, \end{cases}$$

and

$$(4.6)\qquad W_k(ax) = \sum_{m=0}^{\infty}(-)^m\left(\frac{ax}{2}\right)^{k+2m}\frac{[h_{m+k} + h_m]}{m!(k+m)!}.$$

In the above $\gamma = 0.57721 \cdots =$ Euler's constant and

$$(4.7)\qquad h_m = 1 + \tfrac{1}{2} + \cdots + \frac{1}{m}, \quad h_0 = 1.$$

We assume the value of log $(ax/2)$ is known. Then, since expansions for $J_k(ax)$ and $I_k(ax)$ were found in Section 2, and since $N_{k-1}(ax)$ is simply a polynomial in $1/(ax)$, we need expand only the entire part of (4.3), i.e., $W_k(ax)$, in symmetric Jacobi polynomials.

Using the representation (4.6) as $f(x)$ in formula (2.5), a straight-forward derivation gives the series

$$(4.8)\qquad W_k(ax) = \sum_{n=0}^{\infty} A_n P_n^{(\alpha,\alpha)}(x), \quad -1 \leqq x \leqq 1,$$

where

$$(4.9)\qquad A_n = \frac{[(-)^k + (-)^n](n+\alpha+1)_\alpha(n+\alpha+\frac{1}{2})}{2^{n+2\alpha+1}} \cdot\sum_{m=0}^{\infty}\frac{(-)^m(-k-2m)_n}{\left(m+\dfrac{k-n+1}{2}\right)_{n+\alpha+1}}\left(\frac{a}{2}\right)^{k+2m}\frac{[h_{k+m}+h_m]}{m!(k+m)!}.$$

We note that the expansion for $Y_0(ax)$ may also be obtained by partially differentiating (3.2) with respect to v since

$$(4.10)\qquad Y_0(ax) = 2\pi^{-1}\left\{\frac{\partial J_v(ax)}{\partial v}\right\}_{v=0}.$$

A similar procedure yields the expansion for $K_0(ax)$. The Jacobi series for $Y_k(ax)$ and $K_k(ax)$ for $k > 0$, however, are not so easily obtained in this manner.

For $k = 0$ and 1, the Chebyshev cases of (4.3) and (4.4) are

$$(4.11)\quad Y_0(ax) = \frac{2}{\pi}\left[\gamma + \ln\left(\frac{ax}{2}\right)\right] J_0(ax) + \sum_{n=0}^{\infty} E_n\, T_{2n}(x), \qquad 0 < x \leqq 1,$$

$$(4.12)\quad Y_1(ax) = \frac{2}{\pi}\left[\gamma + \ln\left(\frac{ax}{2}\right)\right] J_1(ax) - \frac{2}{\pi a x} + \sum_{n=0}^{\infty} F_n\, T_{2n+1}(x), \quad 0 < x \leqq 1,$$

(4.13) $$K_0(ax) = -\left[\gamma + \ln\left(\frac{ax}{2}\right)\right] I_0(ax) + \sum_{n=0}^{\infty} G_n\, T_{2n}(x), \qquad 0 < x \leqq 1,$$

(4.14) $$K_1(ax) = \left[\gamma + \ln\left(\frac{ax}{2}\right)\right] I_1(ax) + \frac{1}{ax} + \sum_{n=0}^{\infty} H_n\, T_{2n+1}(x), \qquad 0 < x \leqq 1,$$

where

(4.15) $$E_n = \frac{2\epsilon_n \left(\frac{a}{4}\right)^{2n} (-)^{n+1}}{\pi(n!)^2} \sum_{k=0}^{\infty} \frac{(-)^k \left(\frac{a}{2}\right)^{2k} \left(n + \frac{1}{2}\right)_k h_{n+k}}{(n+1)_k (2n+1)_k\, k!},$$

(4.16) $$F_n = \frac{2(-)^{n+1} \left(\frac{a}{4}\right)^{2n+1}}{\pi n!(n+1)!} \sum_{k=0}^{\infty} \frac{(-)^k \left(\frac{a}{2}\right)^{2k} \left(n + \frac{3}{2}\right)_k [h_{n+k+1} + h_{n+k}]}{(n+2)_k (2n+2)_k\, k!},$$

(4.17) $$G_n = \frac{\epsilon_n \left(\frac{a}{4}\right)^{2n}}{(n!)^2} \sum_{k=0}^{\infty} \frac{\left(\frac{a}{2}\right)^{2k} \left(n + \frac{1}{2}\right)_k h_{n+k}}{(n+1)_k (2n+1)_k\, k!},$$

(4.18) $$H_n = -\frac{\left(\frac{a}{4}\right)^{2n+1}}{n!(n+1)!} \sum_{k=0}^{\infty} \frac{\left(\frac{a}{2}\right)^{2k} \left(n + \frac{3}{2}\right)_k [h_{n+k+1} + h_{n+k}]}{(n+2)_k (2n+2)_k\, k!}.$$

5. Tables. Tables 1 through 3 are based on the Chebyshev polynomial cases of the expansions given in the previous sections of this paper. The entries in Tables 1 and 2 were computed on the UNIVAC 1103-A and those in Table 3 on the IBM 7090 at ASD. The calculations were designed so that the error incurred in using the expansions whose coefficients are tabulated here will not exceed five units in the 15th decimal place. Spot checks indicate the error is even less. Because all entries are to 16 significant figures, the expansions may be rearranged in powers of x with no loss of accuracy.

The number in parentheses after each entry is the power of ten by which the entry is to be multiplied. We have chosen coefficients corresponding to $a = 5$, but the coefficients for other values of a from one through ten are available on request.

Note that the expansions in this paper are valid not only for $-1 \leqq x \leqq 1$ but for complex x in a region which can be determined by a theorem of Szegö [1, p. 238]. More specifically, a Jacobi series representing an entire function converges everywhere in the finite complex plane. However, the further x lies away from $-1 \leqq x \leqq 1$, the more the accuracy of the expansion deteriorates. This is so because $P_n^{(\alpha,\alpha)}(x)$ for complex x can no longer be bounded by a simple power of n but behaves in the following manner [10]

(5.1) $$P_n^{(\alpha,\alpha)}(z) = \frac{\Gamma(n+\alpha+1)}{n!\pi^{1/2}} N^{2\gamma} \left(\sin\frac{\theta}{2}\right)^{2\gamma} \left(\cos\frac{\theta}{2}\right)^{-2\gamma-2\alpha-1} \cdot \cos\,[N\theta + \pi\gamma]\left\{1 + 0\left(\frac{1}{N}\right)\right\}$$

valid in the z plane cut from -1 to $-\infty$ and from 1 to ∞. In (5.1), $\cos\theta = z$,

TABLE 1
Coefficients for the Series

$$J_0(x) = \sum_{n=0}^{\infty} A_n T_{2n}(x/5) \qquad J_1(x) = \sum_{n=0}^{\infty} B_n T_{2n+1}(x/5) \qquad I_0(x) = \sum_{n=0}^{\infty} C_n T_{2n}(x/5) \qquad I_1(x) = \sum_{n=0}^{\infty} D_n T_{2n+1}(x/5)$$

$$-5 \leqq x \leqq 5$$

n	A_n	B_n	C_n	D_n
0	2.34098 98253 24576 (−03)	−4.81025 79874 58212 (−02)	1.08230 41593 72444 (+01)	1.65591 83236 43522 (+01)
1	−4.94205 09340 95238 (−01)	−4.43466 65460 22008 (−01)	1.26677 21318 60009 (+01)	6.42500 61815 55151 (+00)
2	3.97937 36723 20755 (−01)	1.93233 13296 91198 (−01)	3.25873 16530 56593 (+00)	1.21103 55366 64603 (+00)
3	−9.38314 58796 59383 (−02)	−3.19623 68142 70534 (−02)	4.50054 56944 84814 (−01)	1.30904 56998 82480 (−01)
4	1.08875 31648 68103 (−02)	2.87773 31330 73935 (−03)	3.80753 97089 75161 (−02)	9.06329 93010 42887 (−03)
5	−7.60626 76577 33766 (−04)	−1.64773 93001 95709 (−04)	2.15738 77227 50612 (−03)	4.33748 41004 04394 (−04)
6	3.56948 36463 56946 (−05)	6.56128 50055 62524 (−06)	8.72062 45418 29624 (−05)	1.51584 32032 61766 (−05)
7	−1.20606 97061 36835 (−06)	−1.92705 34880 37504 (−07)	2.63488 07999 39597 (−06)	4.03099 55295 59207 (−07)
8	3.07903 85720 34403 (−08)	4.35311 98064 51247 (−09)	6.16685 39084 19237 (−08)	8.42090 28170 89711 (−09)
9	−6.15440 55412 06848 (−10)	−7.80521 83217 68401 (−11)	1.14988 29923 31926 (−09)	1.41745 27229 98507 (−10)
10	9.89883 30623 59870 (−12)	1.13848 12811 94913 (−12)	1.74728 40587 55705 (−11)	1.96254 75993 94277 (−12)
11	−1.30938 62877 22900 (−13)	−1.37786 52001 23884 (−14)	2.20433 13796 56194 (−13)	2.27359 85296 82644 (−14)
12	1.44992 54555 46092 (−15)		2.34504 22753 50325 (−15)	

TABLE 2
Coefficients for the Series

$$Y_0(x) = (2/\pi)\{\gamma + ln(x/2)\}J_0(x) + \sum_{n=0}^{\infty} E_n T_{2n}(x/5) \qquad Y_1(x) = (2/\pi)\{\gamma + ln(x/2)\}J_1(x) + \sum_{n=0}^{\infty} F_n T_{2n+1}(x/5) - 2/\pi x$$

$$K_0(x) = -\{\gamma + ln(x/2)\}I_0(x) + \sum_{n=0}^{\infty} G_n T_{2n}(x/5) \qquad K_1(x) = \{\gamma + ln(x/2)\}I_1(x) + \sum_{n=0}^{\infty} H_n T_{2n+1}(x/5) - 1/x$$

$$0 < x \leqq 5$$

n	E_n	F_n	G_n	H_n
0	2.06225 35144 48362 (−01)	6.16723 32064 62446 (−01)	1.44570 70580 38540 (+01)	−2.33572 75478 64823 (+01)
1	−1.66649 68241 18285 (−01)	1.01148 30681 23224 (−01)	1.94363 50195 81009 (+01)	−1.05995 40069 53968 (+01)
2	−2.62525 23404 65465 (−01)	−1.63819 01170 12895 (−01)	5.80235 95441 72479 (+00)	−2.28801 59175 41593 (+00)
3	9.57577 15009 74545 (−02)	3.61543 51643 46677 (−02)	9.01848 87443 62640 (−01)	−2.74798 33365 99269 (−01)
4	−1.34756 13306 42697 (−02)	−3.78893 02501 89891 (−03)	8.36577 27959 36959 (−02)	−2.06655 58368 06775 (−02)
5	1.06032 54643 99260 (−03)	2.39870 11377 45682 (−04)	5.10092 82215 62887 (−03)	−1.05709 50504 93099 (−03)
6	−5.41408 18903 44878 (−05)	−1.02859 55207 54039 (−05)	2.18934 64152 72086 (−04)	−3.30317 93418 52240 (−05)
7	1.95165 72417 48920 (−06)	3.20202 24989 85112 (−07)	6.95536 19902 36087 (−06)	−1.08728 98645 17578 (−06)
8	−5.24918 47596 77735 (−08)	−7.58631 71289 13763 (−09)	1.69908 13404 94038 (−07)	−2.36391 19626 27828 (−08)
9	1.09579 84960 92691 (−09)	1.41602 52452 47205 (−10)	3.28795 17709 40376 (−09)	−4.12016 28736 68884 (−10)
10	−1.82911 13098 12991 (−11)	−2.13823 11419 29658 (−12)	5.16184 89074 51723 (−11)	−5.88302 36091 76335 (−12)
11	2.49885 87269 59892 (−13)	2.66758 44662 51279 (−14)	6.70373 01203 98254 (−13)	−7.00552 44636 66701 (−14)
12	−2.84706 00495 41412 (−15)		7.31980 66017 91276 (−15)	

TABLE 3

Coefficients for the Series

$$x^{-\nu}J_\nu(x) = \sum_{n=0}^{\infty} A_n^{(\nu)}T_{2n}(x/5) \qquad x^{-\nu}I_\nu(x) = \sum_{n=0}^{\infty} B_n^{(\nu)}T_{2n}(x/5)$$

$$-5 \leqq x \leqq 5$$

n	$A_n^{(1/3)}$	$B_n^{(1/3)}$	$A_n^{(-1/3)}$	$B_n^{(-1/3)}$
0	8.94076 67920 14735 (−02)	6.58297 78114 19436 (+00)	−1.07166 32561 37673 (−01)	1.73346 79232 80504 (+01)
1	−4.81921 28594 44027 (−01)	7.20005 50672 62920 (+00)	−2.97571 62513 05484 (−01)	2.16272 09227 74341 (+01)
2	2.59994 46413 53814 (−01)	1.70885 10971 09031 (+00)	5.54730 81817 97741 (−01)	6.04768 89225 73269 (+00)
3	−5.17388 91497 92902 (−02)	2.19489 00443 12230 (−01)	−1.62312 94316 81218 (−01)	9.01519 59809 63211 (−01)
4	5.39598 09479 79882 (−03)	1.74386 20597 80771 (−02)	2.12830 59224 11643 (−02)	8.14844 58153 55754 (−02)
5	−3.48602 79717 45306 (−04)	9.35725 57411 65672 (−04)	−1.62033 34074 85423 (−03)	4.88835 76577 22940 (−03)
6	1.53744 49741 62028 (−05)	3.60628 89498 98151 (−05)	8.12833 49697 02066 (−05)	2.07687 29885 63908 (−04)
7	−4.93335 30371 51078 (−07)	1.04455 75512 24314 (−06)	−2.90085 92814 81608 (−06)	6.55696 66640 79319 (−06)
8	1.20489 92956 96342 (−08)	2.35404 20416 24506 (−08)	7.75818 87529 98852 (−08)	1.59597 72584 67654 (−07)
9	−2.31664 13965 03490 (−10)	4.24192 23858 96914 (−10)	−1.61479 04639 34516 (−09)	3.08285 63507 75112 (−09)
10	3.59939 64540 78735 (−12)	6.24808 74659 21528 (−12)	2.69219 41698 10025 (−11)	4.83729 62983 39093 (−11)
11	−4.61479 50001 45118 (−14)	7.66029 28778 25032 (−14)	−3.67795 65706 02520 (−13)	6.28465 97844 22833 (−13)
12			4.19393 32335 77453 (−15)	6.86947 00674 93498 (−15)

n	$A_n^{(2/3)}$	$B_n^{(2/3)}$	$A_n^{(-2/3)}$	$B_n^{(-2/3)}$
0	1.26821 19075 59155 (−01)	3.89700 09178 94441 (+00)	−1.34833 35738 84306 (−01)	2.70904 14407 24111 (+01)
1	−3.82426 30216 91411 (−01)	3.97522 01852 22248 (+00)	2.53684 77592 21459 (−01)	3.58034 51642 90575 (+01)
2	1.58419 46147 00555 (−01)	8.73294 88291 09913 (−01)	6.72159 99132 26151 (−01)	1.09071 34473 17453 (+01)
3	−2.73985 78534 08571 (−02)	1.04699 27477 53559 (−01)	−2.65206 75332 02099 (−01)	1.76188 79963 65858 (+00)
4	2.59903 60286 58541 (−03)	7.83606 87385 22999 (−03)	4.01486 63047 46445 (−02)	1.70744 74910 71144 (−01)
5	−1.56262 72721 66761 (−04)	3.99168 33880 23007 (−04)	−3.36287 90613 98914 (−03)	1.08765 98567 69296 (−02)
6	6.50257 66818 64264 (−06)	1.46964 86166 29454 (−05)	1.81307 01139 10497 (−04)	4.86813 02946 11161 (−04)
7	−1.98695 79839 64146 (−07)	4.08724 99803 03382 (−07)	−6.85821 46199 38979 (−06)	1.60889 95124 35949 (−05)
8	4.65189 13454 60930 (−09)	8.88083 87568 65976 (−09)	1.92618 60920 91631 (−07)	4.07864 59335 37985 (−07)
9	−8.61663 19643 98472 (−11)	1.54819 86261 88014 (−10)	−4.18242 83196 21301 (−09)	8.17158 76141 59425 (−09)
10	1.29480 93295 22571 (−12)	2.21246 66336 55002 (−12)	7.23806 22406 35861 (−11)	1.32536 13614 40354 (−10)
11	−1.61062 10063 91952 (−14)	2.63810 71950 41641 (−14)	−1.02241 47112 37617 (−12)	1.77479 93677 21115 (−12)
12			1.20165 09941 09705 (−14)	1.99468 46072 70085 (−14)

TABLE 3 (*Continued*)

n	$A_n^{(1/4)}$	$B_n^{(1/4)}$	$A_n^{(-1/4)}$	$B_n^{(-1/4)}$
0	7.21592 24626 52688 (−02)	7.47302 10303 05303 (+00)	−8.17099 31298 82310 (−02)	1.54456 46447 18517 (+01)
1	−4.97105 11867 57523 (−01)	8.31530 14410 19353 (+00)	−3.74456 45549 83537 (−01)	1.89744 72346 22373 (+00)
2	2.91288 83390 42546 (−01)	2.01309 13436 22401 (+00)	5.16008 82005 50960 (−01)	5.19510 68992 00228 (+00)
3	−6.02831 47437 67723 (−02)	2.63206 92238 80861 (−01)	−1.42226 29467 31262 (−01)	7.59501 67795 49208 (−01)
4	6.44904 67659 23062 (−03)	2.12367 20333 36512 (−02)	1.80565 04668 12229 (−02)	6.75005 48548 48750 (−02)
5	−4.24592 56017 28019 (−04)	1.15486 74718 87051 (−03)	−1.34437 14794 53047 (−03)	3.99099 14294 53922 (−03)
6	1.90112 60876 55165 (−05)	4.50337 94565 14874 (−05)	6.62934 97557 47040 (−05)	1.67428 98832 97653 (−04)
7	−6.17793 29918 73178 (−07)	1.31804 25685 19379 (−06)	−2.33304 02865 38664 (−06)	5.22735 04185 32855 (−06)
8	1.52537 19747 09771 (−08)	2.99821 23052 29824 (−08)	6.16613 44735 40195 (−08)	1.25977 09051 28136 (−07)
9	−2.96097 83840 28929 (−10)	5.44849 42083 22499 (−10)	−1.27028 56543 85534 (−09)	2.41177 00010 18902 (−09)
10	4.63993 80496 72974 (−12)	8.08732 15913 15397 (−12)	2.09864 07533 16501 (−11)	3.75370 60782 78502 (−11)
11	−5.99493 04600 18949 (−14)	9.98561 69608 87138 (−14)	−2.84373 93655 71641 (−13)	4.84076 50499 38746 (−13)
12		1.04140 77287 05166 (−15)	3.21872 59210 83734 (−15)	5.25514 33653 87754 (−15)

n	$A_n^{(3/4)}$	$B_n^{(3/4)}$	$A_n^{(-3/4)}$	$B_n^{(-3/4)}$
0	1.29259 76634 12324 (−01)	3.40367 50320 63991 (+00)	−1.07108 00067 60494 (−01)	3.01807 22091 21124 (+01)
1	−3.53485 41224 59797 (−01)	3.41159 09039 09293 (+00)	4.59282 45203 50027 (−01)	4.04154 70990 58814 (+01)
2	1.38693 46692 29917 (−01)	7.35518 91333 63896 (−01)	6.82534 33005 94923 (−01)	1.25813 97685 36136 (+01)
3	−2.32397 67410 71826 (−02)	8.67224 88421 14764 (−02)	−2.96708 11160 24287 (−01)	2.07484 85659 26079 (+00)
4	2.15622 40367 90532 (−03)	6.39730 24996 65543 (−03)	4.67644 95998 05034 (−02)	2.04728 16790 32805 (−01)
5	−1.27441 43552 17310 (−04)	3.21786 27908 36382 (−04)	−4.01877 21906 54069 (−03)	1.32448 04086 74998 (−02)
6	5.22966 96656 56027 (−06)	1.17162 12539 26833 (−05)	2.20820 19108 00076 (−04)	6.00806 99443 67353 (−04)
7	−1.57920 21138 77838 (−07)	3.22621 49048 88015 (−07)	−8.48036 03378 59174 (−06)	2.00909 87697 36448 (−05)
8	3.65942 38094 19545 (−09)	6.94760 17547 32315 (−09)	2.41209 84427 69199 (−07)	5.14647 54652 28697 (−07)
9	−6.71688 65154 82103 (−11)	1.20138 77599 87366 (−10)	−5.29481 93781 41679 (−09)	1.04076 51295 38435 (−08)
10	1.00112 47892 05989 (−12)	1.70415 17978 78482 (−12)	9.25118 40343 21647 (−11)	1.70234 23895 70753 (−10)
11	−1.23610 48693 00485 (−14)	2.01814 93824 57457 (−14)	−1.31797 96261 75435 (−12)	2.29723 86885 18947 (−12)
12			1.56102 37245 93056 (−14)	2.60017 90694 76929 (−14)

$N = [n(n + 2\alpha + 1)]^{1/2}, \gamma = -(1 + 2\alpha)/4$. In general, if values of $f(x)$ for complex x are desired, it is wisest to choose a such that the expansions are interpolatory along a suitable ray in the complex x-plane and to stay as close as possible to this ray.

Suppose we have the truncated expansion

$$f(x) = \sum_{n=0}^{N} A_n T_n(x) + \epsilon_{N+1} = \phi_N(x) + \epsilon_{N+1}, \quad -1 \leqq x \leqq 1, \tag{5.2}$$

and

$$\epsilon_{N+1} = \sum_{n=N+1}^{\infty} A_n T_n(x) \tag{5.3}$$

Then $\phi_N(x)$ is not generally the Chebyshev approximation of degree N to $f(x)$ in the sense of [11], i.e., the polynomial $\Phi_N(x)$ of degree N uniquely characterized by the fact that in the interval $[-1, 1]$ the number of consecutive points at which the difference $f(x) - \Phi_N(x)$ with alternate changes in sign assumes the value

$$\max_{-1 \leqq x \leqq 1} | f(x) - \Phi_N(x) |,$$

is not less than $N + 2$; but $\phi_N(x)$ may closely approximate $\Phi_N(x)$. How closely, of course, depends on the coefficients A_n. If A_n goes quite rapidly to zero as $n \to \infty$, then A_{N+2} is small compared to A_{N+1} and consequently

$$\epsilon_{N+1} \sim A_{N+1} T_{N+1}(x) \tag{5.4}$$

and the error curve is practically uniform, i.e., $\phi_N(x)$ is nearly $\Phi_N(x)$. Such is the case in our expansions, and, consequently, we must expect the approximation $\Phi_N(x)$ for moderate values of a to offer a negligible improvement over the Chebyshev polynomial expansions derived in this paper and truncated after $N + 1$ terms.

6. Acknowledgment. This work was supported by the United States Air Force through the Aeronautical Systems Division (ASD), Wright-Patterson Air Force Base, Ohio. The coefficients tabulated in this paper, as well as many others, were computed there, and the author gratefully acknowledges the programming skills of Messrs. Ralph Graham and Hank Bryson of ASD. Special thanks are due to Yudell Luke for his helpful suggestions and comments.

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