TECHNICAL NOTES AND SHORT PAPERS

On the Numerical Solution of a Differential-Difference Equation Arising in Analytic Number Theory

By R. Bellman and B. Kotkin

Summary. The computational solution of a certain class of differential-difference equations requires numerical procedures involving an extremely high degree of precision to obtain accurate results over a large range of the independent variable. One method of solution uses an iterative procedure which relates the differential-difference equation over a large range to a system of ordinary differential equations over a limited range. When the characteristic roots of the related system indicate borderline stability, it is evident that small perturbations in obtaining successive initial values eventually grow out of control as the system increases.

To investigate this phenomenon, we examine the equation $u'(x) = -u(x-1)/x$. arising in analytic number theory.

1. Introduction. The function $\psi(x, y)$ equal to the number of integers less than or equal to x and free of prime factors greater than y is of obvious interest in number theory. It has been investigated by Chowla and Vijayaraghavan, Ramaswami, Buchstab, and de Bruijn [1], where references to the other works may be found. It has been shown that

$$\lim_{y\to\infty} y^x\psi(y^x, y) = y(x) \tag{1.1}$$

exists, where $y(x)$ is a function satisfying the interesting functional equation

$$y'(x) = \frac{-y(x-1)}{x}, \qquad x > 1, \tag{1.2}$$

with $y(x) = 0, \quad x < 0, \quad y(x) = 1, \quad 0 \leqq x \leqq 1$.

The problem of computing the values of $y(x)$ over an initial range, say $1 \leqq x \leqq 20$, was posed to the authors by M. Hall. Although we possess a method described in [2], [3] which reduces this problem to solving successive systems of differential equations, the foregoing equation is still of interest, since some useful information concerning the accuracy of our technique is obtained.

Tables of $y(x)$ are given which are of eight significant figure accuracy for $1 \leqq x \leqq 5$ and of two or more significant figure accuracy up to $x = 20$. With additional effort, more significant figures could be obtained.

2. Computational Procedure. As indicated in [2], [3], the single equation of (1.2)

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TABLE 1
$H = 2^{-8}$

x	$y(x)$	$y(x+1)$	$y(x+2)$	$y(x+3)$	$y(x+4)$
1.0000	1.00000000	0.30685282	0.48608400 (10^{-1})	0.49109353 (10^{-2})	0.354732 (10^{-3})
1.0625	0.93937538	0.27701857	0.42592619 (10^{-1})	0.42047710 (10^{-2})	0.298211 (10^{-3})
1.1250	0.88221696	0.24983249	0.37274025 (10^{-1})	0.35958222 (10^{-2})	0.250439 (10^{-3})
1.1875	0.82814974	0.22504611	0.32575066 (10^{-1})	0.30712743 (10^{-2})	0.210104 (10^{-3})
1.2500	0.77685644	0.20244167	0.28427227 (10^{-1})	0.26199627 (10^{-2})	0.176087 (10^{-3})
1.3125	0.72806629	0.18182739	0.24769805 (10^{-1})	0.22321537 (10^{-2})	0.147300 (10^{-3})
1.3750	0.68154627	0.16303349	0.21548879 (10^{-1})	0.18993579 (10^{-2})	0.123315 (10^{-3})
1.4375	0.63709451	0.14590910	0.18716429 (10^{-1})	0.16141689 (10^{-2})	0.103045 (10^{-3})
1.5000	0.59453490	0.13031957	0.16229603 (10^{-1})	0.13701261 (10^{-2})	0.860250 (10^{-4})
1.5625	0.55371290	0.11614431	0.14050084 (10^{-1})	0.11615955 (10^{-2})	0.717502 (10^{-4})
1.6250	0.51449219	0.10327493	0.12143537 (10^{-1})	0.98366684 (10^{-3})	0.597899 (10^{-4})
1.6875	0.47675186	0.91613756 (10^{-1})	0.10479153 (10^{-1})	0.83206424 (10^{-3})	0.497791 (10^{-4})
1.7500	0.44038422	0.81072430 (10^{-1})	0.90292372 (10^{-2})	0.70306896 (10^{-3})	0.414080 (10^{-4})
1.8125	0.40529290	0.71570870 (10^{-1})	0.77688576 (10^{-2})	0.59345175 (10^{-3})	0.344148 (10^{-4})
1.8750	0.37139135	0.63036275 (10^{-1})	0.66755423 (10^{-2})	0.50041395 (10^{-3})	0.285780 (10^{-4})
1.9375	0.33860152	0.55402312 (10^{-1})	0.57290097 (10^{-2})	0.42153594 (10^{-3})	0.237107 (10^{-4})

TABLE 2

x	$y(x)$
6	0.196555 (10^{-4})
7	0.87935 (10^{-6})
8	0.3640 (10^{-7})
9	0.458 (10^{-8})
10	0.319 (10^{-8})
11	0.284 (10^{-8})
12	0.258 (10^{-8})
13	0.236 (10^{-8})
14	0.218 (10^{-8})
15	0.202 (10^{-8})
16	0.189 (10^{-8})
17	0.177 (10^{-8})
18	0.166 (10^{-8})
19	0.157 (10^{-8})
20	0.149 (10^{-8})

is replaced by the system of ordinary differential equations

$$
\begin{aligned}
&(0) && y_0'(x) = 0,\\
&(1) && y_1'(x) = \frac{-y_0(x)}{x+1},\\
(2.1)\quad &(j) && y_j'(x) = \frac{-y_{j-1}(x)}{x+j},\\
& && \vdots\\
&(N) && y_N'(x) = \frac{-y_{N-1}(x)}{x+N}.
\end{aligned}
$$

Equations (2.1) (0) and (2.1) (1) with initial values $y_0(0) = 1$, $y_1(0) = 1$ are integrated numerically to obtain the value $y_1(1) = y_2(0)$. The three equations (2.1)(0), (2.1)(1), (2.1)(2) are integrated numerically to obtain the initial value $y_2(1) = y_3(0)$, etc. Since $y_j(x) = y(x + j)$, $j = 0, 1, \cdots, N$, we obtain in this way the solution $y(x)$ in $0 \leqq x \leqq N + 1$.

3. Numerical Results of Digital Computer Experiments. Using an IBM 7090 Fortran program with integration subroutines INT and INTM from the IBM Share Library D2 RWFINT and a fixed grid size H, our results agreed to eight significant figures up to $N = 5$ for $H = 2^{-7}$ and $H = 2^{-8}$. As N increased, the agreement got poorer and at $x = 20$ there was agreement to only three significant figures with the initial value $y_1(0) = 1$.

4. Stability. Examining the related system of differential equations, we note that the characteristic values of the matrix of coefficients are all zero. Consequently, we are on the borderline of stability, and progressive loss of accuracy is to be expected as N increases. Using finer grids and more precise methods, we could, of course, decrease the rate of loss of accuracy. In an earlier computation of the solution of differential-difference equations [3] this effect was not present, and more accurate results were obtained.

5. Tables. In this section we present two tables of values. The first presents values of $y(x)$ at intervals of $\frac{1}{16}$ accurate to six or more significant figures. The second presents subsequent values to the degree of accuracy we possess. Observe that as x increases, the number of significant figures decreases.

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1. N. G. deBruijn, "On the number of positive integers $< x$ and free of prime factors $> y$," *Nederl. Akad. Wetensch. Proc. Ser. A*, v. LIV, n. 1, 1951, p. 1–12.
2. R. Bellman, "On the computational solution of differential-difference equations," *J. Math. Anal. Appl.*, v. 2, 1961, p. 108–110.
3. R. Bellman, & B. Kotkin, *On the Computational Solution of a Class of Nonlinear Differential-Difference Equations*, The RAND Corporation, Paper P-2233, February 1961.

The Coefficients of the Lemniscate Function

By L. Carlitz

Let $\wp(u)$ denote the special Weierstrass $\wp$-function that satisfies the differential equation

$$\wp'^2(u) = 4\wp^3(u) - 4\wp(u).$$

Hurwitz [4] put

$$\wp(u) = \frac{1}{u^2} + \frac{2^4 E_1}{4}\frac{u^2}{2!} + \frac{2^8 E_2}{8}\frac{u^6}{6!} + \cdots + \frac{2^{4n} E_n}{4n}\frac{u^{4n-2}}{4n-2} + \cdots .$$

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