p	Factors of M_p
8167	76835137
8171	9412993
8209	14759783
8221	9667897 · 18480809
8269	19630607
8273	28062017 · 62014409
8287	36877151
8377	134033 · 787439 · 2596871
8429	455167 · 927191
8467	6655063
8539	13662401
8563	32402393
8573	12345121
8623	80504329
8699	43790767
8737	6640121
8741	5926399
8849	52368383
8933	36232249
8969	13345873
9029	25913231
9049	28721527 · 28938703
9059	30293297
9109	49625833
9127	8707159
9137	2704553
9161	86901247
9199	53354201
9221	91841161
9283	29352847 · 34031479 · 41532143
9337	2838449 · 2405633
9403	5735831
9479	48532481
9601	3513967 · 16974569 · 17256487
9643	12362327
9743	34626623
9817	20556799
9829	14075129
9851	3723679
9859	1656313
9883	10436449
9973	7419913 · 10591327 · 19367567

Certain Properties of Pyramidal and Figurate Numbers

By M. Wunderlich

It is well known that despite some extensive computation [1], the only two known solutions to the Diophantine equation

$$a^3 + b^3 + c^3 = 3 \tag{1}$$

are $a = b = c = 1$, and $a = b = 4$, $c = -5$. Professor Aubrey Kempner noted at a number theory seminar at the University of Colorado that these solutions also satisfy the equation

$$a^3 + b^3 + c^3 = a + b + c. \tag{2}$$

Therefore, it is of interest whether or not (2) has solutions other than these two and if so, how many. Since there are so few solutions known to (1), it seemed reasonable to conjecture that there would be only finitely many solutions to (2).

If we change the sign of the third variable and divide through by six, we see that (2) is equivalent to

$$\frac{a^3 - a}{6} + \frac{b^3 - b}{6} = \frac{c^3 - c}{6}$$

or

$$\frac{(a-1)(a)(a+1)}{6} + \frac{(b-1)(b)(b+1)}{6} = \frac{(c-1)(c)(c+1)}{6}.$$

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Numbers of the form

$$(3) \qquad \frac{(a-1)(a)(a+1)}{6}, \qquad a \geqq 2$$

had been studied by the ancients and had been given the name "Pyramidal" because of a curious geometric property which they possess [2]. It is also clear from (3) that we can define the pyramidal numbers, P_n, as follows:

$$P_n = \binom{n+2}{3}, \qquad n \geqq 1.$$

In view of the binomial theorem, they are the numbers in the fourth diagonal row of Pascal's triangle, a fact which makes the numerical computation of these numbers very easy.

Therefore, the conjecture mentioned concerning (2) can be restated as follows: There is only a finite number of solutions to

$$(4) \qquad P_x + P_y = P_z.$$

The purpose of this paper is to indicate how the reasonableness of this conjecture was tested and how an examination of a table of solutions of (4) led to some interesting theoretical results. In particular, S. Chowla was able to prove that there are infinitely many solutions to (4) and hence to (2) [3]. Also, S. Segal could prove that the only solution to

$$2P_x = P_y$$

is where $x = 3$ and $y = 4$ [4].

The computation of the first 88 solutions of (4) was done with the help of V. Keiser on the C.D.C. 1604 digital computer at the National Bureau of Standards laboratories in Boulder, Colorado. The method used was systematically to determine for every pair of pyramidal numbers P_n, P_m, $m < 13{,}000$, $n < m$, whether or not $P_m - P_n$ was again pyramidal. If we let $M = P_m - P_n$ it is necessary to determine whether

$$M = P_v, \quad v = 1, 2, \cdots n - 1.$$

To do this, the following "hunting" procedure was employed: All the numbers $P_1, P_2, \cdots P_n$ were stored in the machine in ascending order of magnitude. M was first compared with $P_{[n/2]}$. ($[n]$ as usual indicates the integral part of n.) If $M = P_{[n/2]}$, a solution to (4) was found. If $M < P_{[n/2]}$, a solution can only exist for $v = 1, 2, \cdots, [n/2] - 1$. If $M > P_{[n/2]}$, a solution can only exist for $v = [n/2] + 1, [n/2] + 2, \cdots, n$. In either case the number of values which v can assume is roughly halved. If this procedure is repeated $[\log m/\log 2] + 1$ times, any solution if it exists will be found. For example, if $m = 15{,}000$, the process need only be repeated 14 times.

Table 1 lists the first 88 solutions found using this program. It required approximately 6 hours of machine time.

It is interesting to note that although S. Chowla proved that there exist infinitely many solutions to (4), he by no means justified the great number of solutions that were found. By imposing extra relations on (2) he was able to reduce it to a Pellian equation which has infinitely many solutions that also satisfy (2). Therefore, he

TABLE 1

x	y	z	x	y	z
3	3	4	1351	1478	1786
8	14	15	798	1818	1868
20	54	55	438	2164	2170
30	55	58	1146	2072	2183
39	70	74	1139	2115	2220
61	102	109	1609	1941	2256
84	90	110	1105	2303	2385
34	118	119	853	2417	2452
48	138	140	1103	2514	2583
119	154	175	1484	2584	2738
187	201	245	1089	2773	2828
100	290	294	834	2958	2980
327	336	418	528	3138	3143
149	429	435	1775	2954	3154
252	424	452	1484	3094	3204
248	450	474	2478	2726	3286
362	415	492	2099	3211	3486
219	515	528	729	3595	3605
136	532	535	2200	3660	3908
424	448	550	742	4415	4422
314	527	562	2116	4580	4726
434	495	588	2948	4408	4810
399	588	644	3138	4630	5068
324	663	688	2912	4838	5167
272	688	702	868	6034	6040
304	695	714	2252	6390	6482
349	713	740	5338	5608	6900
532	643	747	3570	7154	7439
424	705	753	1271	7554	7566
378	790	818	6152	6586	8034
608	754	868	1160	8070	8078
230	903	908	5300	7284	8120
489	869	918	5630	7105	8129
775	950	1098	6340	6788	8280
703	1064	1158	4115	8034	8379
878	1044	1220	4015	8910	9174
968	1001	1241	7104	7847	9442
922	1286	1428	7062	8094	9592
290	1430	1434	2951	10184	10266
367	1436	1444	1328	10568	10575
855	1343	1450	7842	10168	11532
504	1629	1645	7294	10618	11660
897	1621	1708	8274	10149	11725
750	1690	1738	9050	11100	12824

found infinitely many solutions of a very special type, none of which, incidentally, appear in the table.* Two unsolved problems are to find a parametric representation

* (Added in publication) In the March 1961 issue of *Elemente Der Mathematik*, W. Sierpiński has also shown that there are infinitely many such solutions. His proof, however, does yield two of the solutions in the table, namely $x = 8$, $y = 14$, $z = 15$ and $x = 2912$, $y = 4838$, $z = 5167$.

which will give all the solutions to (2) and to find an asymptotic density function analogous to the prime number theorem. This latter problem can be more explicitly stated as follows: Let $\phi(x)$ represent the number of integers $n \leqq x$ such that there exist positive integers a, b such that

$$P_a + P_b = P_n .$$

Does there exist a continuous function $g(x)$ such that $\phi(x) \sim g(x)$? That is,

$$\lim_{x\to\infty} \frac{\phi(x)}{g(x)} = 1.$$

The concept of pyramidal number can be generalized by defining the rth *figurate number of order n* to be the binomial coefficient

$$(5) \qquad f_{n,r} = \binom{r+n-1}{n}.$$

In this notation the pyramidal number P_x is $f_{3,x}$. Work is now in progress to compute possible solutions to

$$f_{n,x} + f_{n,y} = f_{n,z}$$

where $n = 4, 5$, and 6. One might be led to believe that there are only finitely many solutions to (5) for $n = 4$ for the following reason: Whereas for $n = 3$ the equation was reduced to a Pellian equation, if $n = 4$ a similar reduction may result in a set of cubic Diophantine equations to which the Roth theorem may apply. If the reduction could be effected without imposing any further restrictive relations the conjecture would be proved. Preliminary results show that the *only* two solutions to (5) for $n = 4$ and $z < 5264$ are $x = 4$, $y = 4$, $z = 5$; and $x = 129$, $y = 187$, $z = 197$. It is interesting to note that Paul Erdos had once conjectured in a letter to Mr. Chowla that the only solution of

$$2f_{n,x} = f_{n,y}$$

is $x = n$ and $y = n + 1$. As we have seen, S. Segal has affirmatively confirmed this conjecture for $n = 3$.

It was further noted upon examining a decimal print-out of the first 25,000 pyramidal numbers that the last digit repeated itself in a cycle of 20, i.e.,

$$(6) \qquad \text{If} \quad x \equiv y \pmod{20}, \quad \text{then} \quad f_{3,x} \equiv f_{3,y} \pmod{10}.$$

This observation led to the following generalized result:

THEOREM. *If* $k = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_q^{\gamma_q}$, *and* n *is a positive integer then for* $j = 1, 2, \cdots q$ *let*

$$\beta_j = \left[\frac{n}{p_j}\right] + \left[\frac{n}{p_j^2}\right] + \cdots + \left[\frac{n}{p_j^{\alpha_j}}\right]$$

where $p_j^{\alpha_j}$ *is the largest power of* $p_j \leqq n$. i.e. $\alpha_j = [\log n/\log p_j]$. *Finally let* $t = p_1^{\beta_1} p_2^{\beta_2} \cdots p_q^{\beta_q}$. *If* x *and* y *are positive integers such that* $x \equiv y \pmod{tk}$, *then*

$$f_{n,x} \equiv f_{n,y} \pmod{k}$$

(Note that (6) is a special case of the theorem where $n = 3$ and $k = 10$.)

Proof. For $i = 1, 2, \cdots n$, $(x + n - i) \equiv (y + n - i) \pmod{tk}$ so that $(x + n - 1)(x + n - 2) \cdots (x) \equiv (y + n - 1)(y + n - 2) \cdots (y) \pmod{tk}$. Also from the definition of t, $(n!, tk) = t$ since t is the product of highest powers of $p_1, p_2, \cdots p_q$ which are contained in $n!$. So,

$$f_{n,x} = \frac{(x + n - 1) \cdots (x)}{n!} \equiv \frac{(y + n - 1) \cdots (y)}{n!} = f_{n,y} \pmod k$$

which proves the theorem.

It is not asserted in the theorem that tk is the smallest period. In fact, easy examples show that in many cases a value for t can be found which is strictly smaller than the one specified in the theorem. According to *Mathematical Reviews* (v. 20, 1959, Review no. 1653), the smallest period has evidently been found by S. Zabek [5] to be tk where

$$t = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_q^{\alpha_q}.$$

This information may be quite useful in numerically searching for solutions to (5) since these congruences limit the number of solutions that could possibly exist, thereby reducing the amount of machine time needed for the search.

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Note on Osculatory Rational Interpolation†

By Herbert E. Salzer

Abstract. In n-point osculatory interpolation of order $r_i - 1$ at points x_i, $i = 1, 2, \cdots, n$, by a rational expression $N(x)/D(x)$, where $N(x)$ and $D(x)$ are polynomials $\sum a_j x^j$ and $\sum b_j x^j$, we use the lemma that the system (1) $\{N(x_i)/D(x_i)\}^{(m)} = f^{(m)}(x_i)$, $m = 0, 1, \cdots, r_i - 1$, is equivalent to (2) $N^{(m)}(x_i) = \{f(x_i)D(x_i)\}^{(m)}$, $m = 0, 1, \cdots, r_i - 1$, $D(x_i) \neq 0$. This equivalence does not require $N(x)$ or $D(x)$ to be a polynomial or even a linear combination of given functions. The lemma implies that (1), superficially non-linear in a_j and b_j, being the same as (2), is actually linear. For the n-point interpolation problem, the linear system, of order $\sum_{i=1}^{n} r_i$, which might be large, is replaceable by separate linear

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