Note that the corresponding directed graph has the Hamiltonian circuit $1 \to 6 \to 5 \to 3 \to 4 \to 2 \to 7 \to 1$ and so is strongly connected. Hence, for no permutation P does PAP^{-1} reduce. Using the algorithm described in [3] one obtains the permutations $P = (1\ 7\ 2\ 5\ 6\ 4\ 3)$ and $Q = (1)(2\ 3\ 6\ 7\ 5\ 4)$. Applying P and Q to the rows and columns of A, one obtains:

$$PAQ = \left|\begin{array}{cc|ccc|cc} 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 3 & 4 & 0 & 0 & 0 & 0 \\ 3 & 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 4 & 0 & 0 \\ \hline 3 & 0 & 0 & 3 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 5 & 1 \end{array}\right|$$

The authors are indebted to F. Harary for a pre-publication copy of his paper [4].

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Missing Data Correlation Computations

By R. I. Jennrich

In correlation analysis or in any multivariate analysis based on the computation of a correlation or covariance matrix, the applied statistician often runs into the problem of missing data. To avoid complication in computing the correlation matrix, a complete observation vector is often discarded when only one or more of its components are missing. If a correlation matrix is computed by means of a standard electronic computer program, this procedure is often necessary. A large percentage of data may be thrown away when only a small percentage is missing. This note describes a modification in the standard computing scheme which eliminates this waste of data.

Let $x_{n1}, x_{n2}, \cdots, x_{np}$ denote the p components of the nth observation vector, $n = 1, 2, \cdots, N$. It is customary to add an $n + 1$st component to this vector which is identically equal to one. That is $x_{n,p+1} = 1$. The cross product matrix

$$a_{ij} = \sum_{n=1}^{N} x_{ni}x_{nj} \qquad i = 1, \cdots, p + 1; \quad j = 1, \cdots, p + 1$$

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is customarily computed and the correlation matrix is computed from this matrix by using the fact that

$$a_{p+1,p+1} = N, \; a_{i,p+1} = \sum_{n=1}^{N} x_{ni} \; ; \quad i = 1, \cdots, p.$$

In addition to adding a component which is identically one to each observation vector, let us form a new vector $c_{n1}, c_{n2}, \cdots, c_{np}$ where c_{ni} is zero if the ith component of the observation vector is missing and one otherwise. Letting each element of missing data have value zero, we form the cross product matrices

$$s_{ij} = \sum_{n=1}^{N} x_{ni}x_{nj} \qquad i, j = 1, \cdots, p + 1$$

$$n_{ij} = \sum_{n=1}^{N} c_{ni}c_{nj} \qquad i, j = 1, \cdots, p.$$

The means m_i, covariances v_{ij}, and correlations r_{ij} are computed from these matrices by the formulas

$$m_i = \frac{1}{n_{ii}} s_{i,p+1}$$

$$v_{ij} = \frac{1}{n_{ij}} s_{ij} - m_i m_j$$

$$r_{ij} = \frac{v_{ij}}{\sqrt{v_{ii}}\sqrt{v_{jj}}}.$$

It should be noted that the statistical properties of these estimates will differ slightly from those computed without missing data. A discussion of some of these properties is given by S. S. Wilks [1].

A FORTRAN program for the computations described in this note is in use at the University of Wisconsin. A write-up and program deck can be obtained by writing to the author.

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Polynomial Approximations to $I_0(x)$, $I_1(x)$ and Related Functions

By F. D. Burgoyne

Hitchcock [1] gives polynomial approximations to some Bessel functions of order zero and one and to some related functions. Notable omissions from his list are any approximations to $I_0(x)$ or $I_1(x)$. The following approximations may serve to fill this gap.

If we write $I_n(x) = (2\pi x)^{-1/2} e^x F_n(x)$, then with the maximum error stated in brackets in each case, and provided $0 \leqq t \leqq 1$,

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