$$
\begin{aligned}
I_0(4t) &= 0.99999\ 99985 + 4.00000\ 01935\, t^2 + 3.99999\ 59541\, t^4 \\
&\quad + 1.77780\ 99690\, t^6 + 0.44431\ 89384\, t^8 + 0.07137\ 58187\, t^{10} \\
&\quad + 0.00759\ 42968\, t^{12} + 0.00082\ 67816\, t^{14}\ (17 \times 10^{-10}), \\
t^{-1} I_1(4t) &= 1.99999\ 99997 + 4.00000\ 00421\, t^2 + 2.66666\ 57853\, t^4 \\
&\quad + 0.88889\ 59049\, t^6 + 0.17775\ 04042\, t^8 + 0.02376\ 15011\, t^{10} \\
&\quad + 0.00219\ 03549\, t^{12} + 0.00020\ 11611\, t^{14}\ (4 \times 10^{-10}), \\
(2\pi)^{-1/2} F_0(4/t) &= 0.39894\ 22809 + 0.01246\ 67783\, t + 0.00176\ 23668\, t^2 \\
&\quad + 0.00026\ 22220\, t^3 + 0.00225\ 85672\, t^4 - 0.01283\ 14822\, t^5 \\
&\quad + 0.04958\ 11198\, t^6 - 0.12099\ 40805\, t^7 + 0.18954\ 76618\, t^8 \\
&\quad - 0.18677\ 83276\, t^9 + 0.11133\ 15511\, t^{10} - 0.03666\ 94167\, t^{11} \\
&\quad + 0.00512\ 46015\, t^{12}\ (7 \times 10^{-10}), \\
(2\pi)^{-1/2} F_1(4/t) &= 0.39894\ 22799 - 0.03740\ 06642\, t - 0.00293\ 14981\, t^2 \\
&\quad - 0.00043\ 77220\, t^3 - 0.00237\ 87859\, t^4 + 0.01319\ 50213\, t^5 \\
&\quad - 0.05078\ 72951\, t^6 + 0.12301\ 43060\, t^7 - 0.19083\ 32956\, t^8 \\
&\quad + 0.18552\ 23758\, t^9 - 0.10862\ 98349\, t^{10} + 0.03497\ 54315\, t^{11} \\
&\quad - 0.00474\ 86397\, t^{12}\ (8 \times 10^{-10}).
\end{aligned}
$$

The first two approximations were obtained by the economization method of Lanczos [2], which is used by Hitchcock. As he notes, this method is inapplicable for the last two approximations, and these were obtained by collocation at the zeros of $T_{13}^*(x) = \cos\{13 \cos^{-1}(2x - 1)\}$.

Battersea College of Technology
London, S.W. 11.

1. A. J. M. Hitchcock, "Polynomial approximations to Bessel functions of order zero and one and to related functions," *MTAC*, v. 11, 1957, p. 86–88.
2. C. Lanczos, *Applied Analysis*, Prentice Hall, Inc., New Jersey, 1956.

A Note on the Curve Fitting of Discrete Data by Economization

By F. D. Burgoyne

Suppose that we are given a set of points (x_i, y_i) $0 \leqq i \leqq n$ and we desire to find the polynomial $p(x)$ of given degree $m(<n)$ such that $\max_i | y_i - p(x_i) |$ is a minimum. It is well known that this may be performed in good approximation by using the method of least squares to find the polynomial $q(x)$ of degree m such that $\sum_i \{y_i - q(x_i)\}^2$ is a minimum, and then taking $p(x) = q(x) + c$, where c is constant given by

$$2c = \min_i \{y_i - q(x_i)\} + \max_i \{y_i - q(x_i)\}.$$

Received July 13, 1961.

While several methods (mainly of an iterative nature) exist for finding a better approximation to $p(x)$, e.g., that of Hastings [1], none seem quite so simple as the above. However, the author has for some time also used the following procedure which is both simple and straightforward and does not involve iteration; it is in fact basically the analogue for discrete data of the usual economization process. First, the polynomial of degree n is found which passes exactly through the points (x_i, y_i), then this polynomial is economized to a polynomial $r(x)$ of degree m over the range $\min_i x_i \leqq x \leqq \max_i x_i$. We then take $p(x) = r(x) + d$, where d is a constant given by

$$2d = \min_i \{y_i - r(x_i)\} + \max_i \{y_i - r(x_i)\}.$$

When n is not too large and the x_i are equally spaced $r(x)$ may be found by hand as follows. First the x_i are transformed so that $x_i = i/n$. Now by Newton's interpolation formula the polynomial of degree n which passes exactly through the points (x_i, y_i) is $\sum_i \binom{nx}{i} \Delta^i x_0$, and the polynomial $r_{nim}(x)$, which $\binom{nx}{i}$ becomes when economized to degree m over the range $0 \leqq x \leqq 1$, may be found from a previously prepared table. Then $r(x) = \sum_i r_{nim}(x) \Delta^i x_0$. If desired, a different interpolation formula may be used.

In the vast majority of examples tried by the author it was found that

$$\tfrac{1}{2}\max_i | y_i - q(x_i) - c | \leqq \max_i | y_i - r(x_i) - d | \leqq \max_i | y_i - q(x_i) - c | .$$

In each of the few cases in which $\max_i | y_i - r(x_i) - d |$ was greater than $\max_i | y_i - q(x_i) - c |$ it was found that the points (x_i, y_i) gave an inadequate picture of the polynomial of degree n which passed exactly through them; this rarely occurred when the x_i were equally spaced.

For some of the examples $p(x)$ also was found by using Hastings' method [1]. In each of these cases $\max_i | y_i - r(x_i) - d |$ was less than $\frac{5}{4}\max_i | y_i - p(x_i) |$.

As an illustration consider the following example. We are given the points (0, 1), (0.25, 1), (0.5, 15), (0.75, 79), (1, 253), and we desire the quadratic polynomial which best approximates them. In this case $p(x)$ can be found exactly, and we obtain the following results:

$$p(x) = 17 - 244x + 464x^2, \quad \text{maximum error } 16;$$

$$r(x) + d = 15.5 - 228x + 448x^2, \quad \text{maximum error } 17.5.$$

The maximum error associated with $q(x) + c$ is 19.2.

It may be added that on Pegasus both the least squares and the economization fitting processes take roughly the same time and use roughly the same number of storage locations.

Battersea College of Technology,
London, S.W.11.

1. C. Hastings, *Approximations for Digital Computers*, Princeton University Press, 1955.