Deterministic Simulation of Random Processes

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1. Introduction. For many problems in engineering, economics, mathematics, and the sciences we are required to simulate random processes. The simulation is usually effected by a computer program which generates a non-random, deterministic sequence of numbers $x_1, x_2, \cdots$ which is supposed to resemble a sequence of independent, random samples from the uniform probability distribution on the interval $0 \leqq x < 1$. The purpose of this paper is to define some general properties of random sequences and to investigate certain deterministic sequences which have some or all of these properties. We shall ignore the limitation that a digital computer with a finite word-length and a finite memory, operating under a single stored program, can produce only sequences of limited precision which are ultimately periodic. This limitation is a kind of round-off error. We shall take as a model of a deterministic mechanism any of the stored-program digital computers now commonly used for scientific computation modified in a single respect: let the word-length be infinite; let rational and irrational numbers x be recorded and computed with perfect precision.

The fundamental problem approached in this paper is *to construct an infinite, deterministic sequence x_n which has every property shared by all infinite, random sequences of independent samples from the uniform distribution.*

Equidistribution is a first requirement of randomness. The sequence x_n is equidistributed in $0 \leqq x < 1$ if, for $0 \leqq a < b \leqq 1$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{a \leqq x_n < b \\ 1 \leqq n \leqq N}} 1 = b - a. \tag{1.1}$$

H. Weyl [1] showed that the fractional parts $x_n = \{n\alpha\}$ are equidistributed for any irrational α. A summary of results on equidistribution is given by J. F. Koksma [4]. A sequence in r dimensions $z^{(n)} = (z_1{}^{(n)}, z_2{}^{(n)}, \cdots, z_r{}^{(n)})$ is equidistributed in the unit cube

$$C_r : 0 \leqq z_1 < 1, \qquad 0 \leqq z_2 < 1, \qquad \cdots, \qquad 0 \leqq z_r < 1$$

if, for $0 \leqq a_i < b_i \leqq 1 \quad (i = 1, \cdots, r)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{a_i \leqq z_i{}^{(n)} < b_i (i=1,\cdots,r) \\ 1 \leqq n \leqq N}} 1 = \prod_{i=1}^{r} (b_i - a_i). \tag{1.2}$$

It was shown by Weyl [1] and by van der Corput [2] that the sequence $z^{(n)}$ is equidistributed if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \exp 2\pi i(k_1 z_1{}^{(n)} + k_2 z_2{}^{(n)} + \cdots + k_r z_r{}^{(n)}) = 0 \tag{1.3}$$

for every set of integers $k_1, \cdots, k_r$ not all zero.

Received January 16, 1962.

We require a definition of "probability" for deterministic sequences x_n. Let S_n be a sequence of statements about the numbers x_n. We define

$$\Pr(S_n) = \lim_{N\to\infty} \frac{1}{N} \sum_{\substack{S_n \text{ true} \\ 1 \leqq n \leqq N}} 1 \tag{1.4}$$

when this limit exists. For example, we define

$$\Pr(x_n > x_{n+1}) = \lim_{N\to\infty} \frac{1}{N} \sum_{\substack{x_n > x_{n+1} \\ 1 \leqq n \leqq N}} 1 \tag{1.5}$$

if the limit exists. As another example, definition (1.1) states that a sequence x_n is equidistributed in C_1 if $\Pr(a \leqq x_n < b) = b - a$ for $0 \leqq a < b \leqq 1$.

There are $k!$ possible orderings of k distinct numbers $z_1, z_2, \cdots, z_k$. Correspondingly, there are $k!$ classes O_j $(j = 1, \cdots, k!)$ of vectors $(z_1, \cdots, z_k)$. For example, if $k = 2$ we say $(z_1, z_2) \in O_1$ if $z_1 > z_2$, but $(z_1, z_2) \in O_2$ if $z_2 > z_1$. For a given sequence x_n in C_1 define the k-dimensional vectors

$$z^{(n)} = (x_n, x_{n+1}, \cdots, x_{n+k-1}) \qquad (n = 1, 2, \cdots). \tag{1.6}$$

We shall say that the sequence x_n is *equipartitioned by k's* if

$$\Pr(z^{(n)} \in O_j) = \frac{1}{k!} \qquad (j = 1, \cdots, k!). \tag{1.7}$$

The sequence x_n will be called *equidistributed by k's* if the k-dimensional sequence $z^{(n)}$ is equidistributed in the unit cube C_k. The sequence x_n is *completely equidistributed* if it is equidistributed by k's for every k.

More generally, the sequence of r-dimensional vectors

$$y^{(n)} = (y_1^{(n)}, \cdots, y_r^{(n)}) \qquad (n = 1, 2, \cdots) \tag{1.8}$$

is defined to be *equidistributed by k's* if the sequence of $k \cdot r$ dimensional vectors

$$w^{(n)} = (y_1^{(n)}, \cdots, y_r^{(n)}, y_1^{(n+1)}, \cdots, y_r^{(n+1)}, \cdots, y_1^{(n+k-1)}, \cdots, y_r^{(n+k-1)})$$

is equidistributed in C_{kr}. The sequence $y^{(n)}$ is *completely equidistributed* if it is equidistributed by k's for every k.

For one-dimensional equidistributed sequences x_n we define the *autocorrelation function* $R(\tau)$ and the *spectral density* $\phi(\omega)$ by

$$\begin{aligned} R(\tau) &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(x_n - \frac{1}{2}\right)\left(x_{n+\tau} - \frac{1}{2}\right) \qquad (\tau = 0, 1, \cdots) \\ \phi(\omega) &= R(0)\omega + 2 \sum_{\tau=1}^{\infty} R(\tau) \cos 2\pi\tau\omega \end{aligned} \tag{1.9}$$

if these limits exist. D. L. Jagerman [5] has proved that, if the limit

$$F(\tau, k, \nu) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \sin 2\pi k x_j \sin \nu x_{j+\tau} \tag{1.10}$$

exists, then the equidistributed sequence x_n has the autocorrelation function

$$R(\tau) = \sum_{k=1}^{\infty} \sum_{\nu=1}^{\infty} \frac{F(\tau, k, \nu)}{\pi^2 k \nu}. \tag{1.11}$$

The numbers x_n form a *white sequence* if $R(\tau) = 0$ for all $\tau \neq 0$. Using the theorem (1.11) Jagerman proved that $x_n = \{n^2\alpha\}$ is white.

Finally, for the purpose of generating multi-dimensional sequences from one-dimensional sequences, we define the *r-dimensional derived sequence* $y^{(n)}$ related to the one-dimensional sequence x_n by the formula

$$y^{(n)} = (x_{nr}, x_{nr+1}, \cdots, x_{nr+r-1}) \qquad (n = 1, 2, \cdots).$$

All of the properties equipartition, equidistribution, and whiteness are of interest because they are properties of the truly random sequences which we are trying to simulate. In this paper these properties are studied with regard to several classes of equidistributed sequences. We consider briefly the Weyl sequence $x_n = \{n\alpha\}$, α irrational. We then consider Multiply sequences, which are formed from recursion formulas

$$x_{n+1} = \{Nx_n + \theta\} \qquad (n = 0, 1, \cdots) \tag{1.12}$$

where $N =$ integer > 1. These sequences have a long history in the literature and practice of computation; some early references are given by O. Taussky and J. Todd [6]. These sequences were shown to be equidistributed for almost all x_0 in [7]. Next we discuss "polynomial" sequences.

$$x_n = \{n^p\alpha + c_1 n^{p-1} + c_2 n^{p-2} + \cdots + c_p\}, \quad \alpha \text{ irrational.} \tag{1.13}$$

H. Weyl [1] showed that these sequences are equidistributed. Finally we discuss the sequences $x_n = \{\theta^n\}$, which were shown by J. F. Koksma [3] to be equidistributed for almost all $\theta > 1$.

Summary of results: The Weyl sequence $x_n = \{n\alpha\}$, α irrational, is not equipartitioned by twos.

A Multiply sequence may fail to be equidistributed even if x_0 is transcendental.

Every equidistributed Multiply sequence is equipartitioned by twos.

Every sequence equidistributed by k's is equipartioned by k's.

Let $x_1, x_2, \cdots$ be any sequence in C_1 generated by a recurrence formula $x_{n+1} = F(x_n)$. This sequence cannot be equidistributed by k's for any $k > 1$ if $F(x)$ has any point of continuity in $0 < x < 1$. In particular, no Multiply sequence is equidistributed by k's for any $k > 1$.

Let $x_1, x_2, \cdots$ be an equidistributed sequence satisfying $x_{n+1} = \{Nx_n\}$, $N =$ integer > 1. Then

$$\Pr(x_n > x_{n+1} > x_{n+2}) = \tfrac{1}{6}(1 + N^{-1}).$$

Thus the sequence is not equipartitioned by k's for any $k \geqq 3$.

Let θ and x_0 be fixed. For each $N = 2, 3, \cdots$ form the Multiply sequence $x_n = x_n(N)$ from the recurrence formula (1.12). For almost all x_0 these sequences are equidistributed, and they are asymptotically completely equidistributed in this sense: For every positive integer k

$$\lim_{N\to\infty} \Pr(a_r \leqq x_{n+r}(N) < b_r \quad \text{for} \quad r = 0, \cdots, k-1) = \prod_{r=0}^{k-1} (b_r - a_r)$$

if $0 \leqq a_r < b_r \leqq 1 \qquad (r = 0, \cdots, k-1)$.

For almost all x_0 the Multiply sequence $x_n(N)$ is asymptotically completely equipartitioned in this sense: For every positive integer k

$$\lim_{N\to\infty} \Pr\,(z^{(n)}(N) \in O_j) = \frac{1}{k!} \qquad (j = 1, \cdots, k!)$$

where O_j is any of the $k!$ partitions of C_k, and where $z^{(n)} = (x_n, x_{n+1}, \cdots, x_{n+k-1})$.

Let $f(z_1, \cdots, z_k)$ be any Riemann-integrable function in C_k for which all the one-dimensional Riemann integrals over line segments in C_k exist. Then for almost all starting values x_0 the Multiply sequences $x_n(N)$ have the limiting property

$$\lim_{N\to\infty}\lim_{M\to\infty} \frac{1}{M}\sum_{n=1}^{M} f(x_n(N), \cdots, x_{n+k-1}(N)) = \int_0^1 \cdots \int_0^1 f(z_1, \cdots, z_k)\, dz_1 \cdots dz_k .$$

For almost all x_0 the Multiply sequence defined by (1.12) has the autocorrelation function

$$R(\tau) = N^{-\tau}(\tfrac{1}{12} - \tfrac{1}{2}\{\beta\} + \tfrac{1}{2}\{\beta^2\}) \qquad (\tau = 0, 1, \cdots)$$

where $\beta = (N^\tau - 1)\theta/(N - 1)$. Thus

$$-\tfrac{1}{24}N^{-\tau} \leqq R(\tau) \leqq \tfrac{1}{12}N^{-\tau} \qquad (\tau = 0, 1, \cdots)$$

so that $R(\tau) \to 0$ as $N \to \infty$ uniformly in τ for $\tau \neq 0$; in this sense the Multiply sequence $x_n(N)$ is asymptotically white.

Let $q(x)$ be a polynomial with real coefficients. Suppose that for some x_0 the sequence $x_1, x_2, \cdots$ generated by

$$x_{n+1} = \{q(x_n)\} \qquad (n = 0, 1, \cdots)$$

is equidistributed in $0 \leqq x < 1$. Then either $q(x) = x + \alpha$, α irrational, or $q(x) = Nx + \theta$, $\pm N =$ integer > 1.

The "polynomial" sequence (1.13) of degree p is equidistributed by k's if and only if $k \leqq p$.

Every polynomial sequence of degree $p \geqq 2$ is white.

The sequence $x_n = \{n^2\alpha\}$, α irrational, $0 < \alpha < 1$, is equipartitioned by threes if and only if α is one of the four numbers $(3 \pm \sqrt{3})/12$, $(9 \pm \sqrt{3})/12$.

The sequence $x_n = \{\theta^n\}$ is completely equidistributed for almost all $\theta > 1$.

If $x_n = \{\theta^n\}$ is equidistributed by r's, then θ cannot be an algebraic number of degree $< r$. In particular, if $\{\theta^n\}$ is completely equidistributed, then θ is transcendental.

Every completely equidistributed sequence is white.

There is an equidistributed white sequence x_n for which $\Pr(x_n > x_{n+1})$ is not equal to 1/2.

There is a sequence $x_1, x_2, \cdots$ equidistributed by twos for which the two-dimensional derived sequence $(x_2, x_3), (x_4, x_5), \cdots$ is not equidistributed.

For any Multiply sequence x_n the r-dimensional derived sequence $y^{(n)} = (x_{nr}, x_{nr+1}, \cdots, x_{nr+r-1})$ is not equidistributed in C_r for any $r > 1$.

The r-dimensional derived sequence $y^{(n)}$ formed from the polynomial sequence $x_n = \{\alpha n^p + \cdots + c_p\}$ (α irrational) is equidistributed by k's if and only if $kr \leqq p$.

For almost all $\theta > 1$, for every $r = 1, 2, \cdots$, the r-dimensional derived sequence $y^{(n)}$ formed from $x_n = \{\theta^n\}$ is completely equidistributed.

2. Weyl Sequences. Weyl showed [1] that $x_n = \{n\alpha\}$ is equidistributed if α is irrational. Let us compute $\Pr(x_n > x_{n+1})$. We have $x > \{x + \alpha\}$ when $1 - \alpha \leqq x < 1$. Therefore, by equidistribution,

$$\Pr(x_n > x_{n+1}) = \Pr(1 - \alpha \leqq x_n < 1) = \alpha \neq \tfrac{1}{2}. \tag{2.1}$$

As D. L. Jagerman has shown [5], Weyl sequences are not white. He has shown that, according to definition (1.9),

$$R(\tau) = \frac{1}{12} - \int_0^{\alpha\tau} \left(\frac{1}{2} - \{u\}\right) du \qquad (\tau = 0, 1, 2, \cdots).$$

Since the integral $\int_0^x (\tfrac{1}{2} - \{u\})\, du$ is a periodic function of x, we may also write

$$\begin{aligned} R(\tau) &= \frac{1}{12} - \int_0^{\{\alpha\tau\}} \left(\frac{1}{2} - u\right) du \qquad (\tau = 0, 1\ 2, \cdots) \\ &= \frac{1}{12} - \frac{1}{2}\{\alpha\tau\} + \frac{1}{2}\{\alpha\tau\}^2. \end{aligned}$$

Since the values $\{\alpha\tau\}$ are equidistributed in $(0, 1)$, the sequence $R(\tau)$ takes values dense in the interval $(-\tfrac{1}{24}, \tfrac{1}{12})$ for arbitrarily large integers τ. Therefore, the spectral density $\phi(\omega)$ defined by (1.9) does not exist.

3. Multiply Sequences. Although Multiply sequences, defined by (1.2) with N = integer > 1, are equidistributed for almost all x_0 (c.f. [7]), it is not sufficient for equidistribution that x_0 be irrational.

THEOREM 1. *A Multiply sequence may fail to be equidistributed even if x_0 is transcendental.*

Proof. Let $\theta = 0$, and let x_0 be the Liouville number

$$x_0 = \sum_{\nu=1}^{\infty} N^{-\nu!}.$$

This number is easily shown [9] to be transcendental. Then all the numbers x_n have N-ary expansions beginning with .1 or with .0; in every case

$$x_n = \{N^n x_0\} < N^{-1} + N^{-2} + N^{-3} < 1.$$

Therefore, these numbers fail to be equidistributed.

Incidentally, it was Borel who first proved, by probabilistic arguments, that for almost all positive numbers $x_0 < 1$ the digits $0, \cdots, N - 1$ appear with equal likelihood $1/N$. The proof of equidistribution for $\theta \neq 0$ which appears in [7] follows from the Riesz ergodic theorem.

THEOREM 2. *Every equidistributed Multiply sequence is equipartitioned by 2's.*

Proof. We must show that the x_n are distinct and that $\Pr(x_{n+1} > x_n) = 1/2$ in the sense of the definition of "probability" given by (1.4).

The numbers x_n are distinct because otherwise the sequence would ultimately be periodic and therefore not equidistributed. In other words, if $x_p = x_q$, then $x_{p+i} = x_{q+i}$ for all $i > 0$.

Let G be the set of numbers x in $(0, 1)$ with the property that $\{Nx + \theta\} > x$. Since the x_n are equidistributed, with $x_{n+1} = \{Nx_n + \theta\}$, the theorem will be proved if it is shown that G consists of a finite number of intervals whose lengths total $1/2$. Without loss of generality assume $0 \leqq \theta < 1$. Then for $k = 0, 1, \cdots, N$

$$\{Nx + \theta\} = Nx + \theta - k \quad \text{for} \quad x \text{ in } E_k \tag{3.1}$$

where $E_0, \cdots, E_N$ are the subintervals

$$\begin{aligned} &E_0 : 0 \leqq x < \frac{1-\theta}{N} \\ &E_k : \frac{k-\theta}{N} \leqq x < \frac{k+1-\theta}{N} \qquad (k = 1, \cdots, N-1) \\ &E_N : \frac{N-\theta}{N} \leqq x < 1. \end{aligned} \tag{3.2}$$

Let G_k be $G \cap E_k$, i.e., that portion of G which lies in E_k. In G_k we have $Nx + \theta - k > x$, or $x > (k - \theta)/(N - 1)$. Since $(N - \theta)/(N - 1) > 1$, the set G_N is empty; for $k < N$ the sets G_k are the intervals

$$\begin{aligned} &G_0 : 0 \leqq x < \frac{1-\theta}{N} \\ &G_k : \frac{k-\theta}{N-1} < x < \frac{k+1-\theta}{N} \qquad (k = 1, \cdots, N-1) \end{aligned} \tag{3.3}$$

since

$$\frac{k-\theta}{N} < \frac{k-\theta}{N-1} \leqq \frac{k+1-\theta}{N} \qquad (k = 1, \cdots, N-1).$$

Therefore, G is the union of the intervals $G_0, \cdots, G_{N-1}$, whose lengths total

$$|G| = \sum_{k=0}^{N-1} |G_k| = \frac{1-\theta}{N} + \sum_{k=1}^{N-1} \left(\frac{k+1-\theta}{N} - \frac{k-\theta}{N-1} \right) \tag{3.4}$$

for which an elementary computation gives the value $|G| = 1/2$. This completes the proof.

This is a convenient context in which to prove the general result:

THEOREM 3. *Every sequence equidistributed by k's is equipartitioned by k's.*

Proof. Given a sequence x_n in the interval C_1 such that the vectors $z^{(n)} = (x_n, x_{n+1}, \cdots, x_{n+k-1})$ are equidistributed in C_k, we must show that these vectors lie in the set O_j with probability $1/k!$ where O_j is any one of the $k!$ subsets of C_k:

$$O_1 : z_1 < \cdots < z_k, \cdots, \qquad O_{k!} : z_k < \cdots < z_1 . \tag{3.5}$$

This is an immediate consequence of the well-known fact (see, for example, Koksma [4]) that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(z^{(n)}) = \int_0^1 \cdots \int_0^1 f(z)\, dz_1 \cdots dz_k \tag{3.6}$$

for all Riemann-integrable functions $f(z) = f(z_1, \cdots, z_k)$ if the sequence $z^{(n)}$ is equidistributed in C_k. In our case we simply define $f(z) = 1$ if z is in O_j, $f(z) = 0$ otherwise. Then formula (3.6) reads:

$$\Pr(z^{(n)} \text{ in } O_j) = \text{volume of } O_j. \tag{3.7}$$

By symmetry O_j has volume $1/k!$.

Returning to Multiply sequences, we observe from the following general result that no Multiply sequence is equidistributed by k's for any $k > 1$.

THEOREM 4. *Let $x_1, x_2, \cdots$ be any sequence in C_1 generated by a recurrence formula $x_{n+1} = F(x_n)$. This sequence cannot be equidistributed by k's for any $k > 1$ if $F(x)$ has any point of continuity in $0 < x < 1$.*

Proof. Let $F(x)$ be continuous at $x = a$, with $F(a) = b$. Then there is a number $\delta > 0$ such that $|F(x) - b| < 1/4$ if $|x - a| < \delta$. Let I be the intersection of $|x - a| < \delta$ with $0 \leqq x < 1$; let J be any interval in C_1 whose distance from b exceeds $1/4$. Then no points (x_n, x_{n+1}) lie in the rectangle $I \times J$. Therefore, the sequence x_n is not equidistributed by twos, hence not equidistributed by k's for any $k > 1$.

In the case of Multiply sequences, $F(x) = \{Nx + \theta\}$, which is continuous at all but N or $N - 1$ points in $(0, 1)$.

The question remains: Are Multiply sequences at least equipartitioned by k's for all k? We answer this question in the case $\theta = 0$.

THEOREM 5. *Let $x_1, x_2, \cdots$ be an equidistributed sequence satisfying $x_{n+1} = \{Nx_n\}$, where $N =$ integer > 1. Then*

$$\Pr(x_n > x_{n+1} > x_{n+2}) = \tfrac{1}{6}(1 + N^{-1}). \tag{3.8}$$

Thus, the sequence x_n is not equipartitioned by k's for any $k \geqq 3$.

Proof. For $0 \leqq x < 1$ define

$$y = \{Nx\}, \qquad z = \{Ny\} = \{N^2x\}. \tag{3.9}$$

Let G be the set of x such that $x > y > z$. Since x_n is equidistributed, it will suffice to show that G is a collection of a finite number of intervals whose lengths total $(1 + N^{-1})/6$. For this purpose we use the Borel interpretation of y and z. We have

$$x = \frac{A}{N} + \frac{y}{N}, \qquad y = \frac{B}{N} + \frac{z}{N} \tag{3.10}$$

where A and B are integers between 0 and $N - 1$, and where x, y, and z are $\geqq 0$ and <1. We proceed to enumerate the cases in which $x > y > z$. If $A = 0$, then $x = y/N \leqq y$, so that the relation $x > y > z$ is impossible; in the same way we conclude that $B > 0$. If $A = 1, \cdots, N - 1$ we have $x > y$ when

$$\frac{A}{N} + \frac{y}{N} > y, \quad \text{or} \quad y < \frac{A}{N-1}. \tag{3.11}$$

Similarly, $y > z$ when $z < B/(N - 1)$. But $x > y$ implies $A \geqq B$, since

$$x = \frac{A}{N} + \frac{y}{N} < \frac{A+1}{N} \leqq y \quad \text{if} \quad B \geqq A + 1. \tag{3.12}$$

Therefore $x > y > z$ implies

$$N - 1 \geqq A \geqq B \geqq 1, \qquad 0 \leqq z < B/(N - 1). \tag{3.13}$$

Conversely, these inequalities imply

$$(3.14)\qquad y = \frac{B}{N} + \frac{z}{N} < \frac{A}{N} + \frac{1}{N} \cdot \frac{A}{N-1} = \frac{A}{N-1}.$$

Therefore, by (3.11), the inequalities (3.13) imply $x > y$. Since the second half of (3.13) implies $y > z$, the inequalities (3.13) are sufficient as well as necessary for $x > y > z$. But each x in C_1 has a unique N-ary representation

$$(3.15)\qquad x = \frac{A}{N} + \frac{B}{N^2} + \frac{z}{N^2}.$$

Therefore, by (3.13) we see that the set $G = (x|x > y > z)$ is a collection of a finite number of intervals whose lengths total

$$(3.16)\qquad \begin{aligned} |G| &= \sum_{A=1}^{N-1} \sum_{B=1}^{A} \frac{1}{N^2} \cdot \frac{B}{N-1} \\ &= \frac{1}{N^2(N-1)} \sum_{A=1}^{N-1} \frac{1}{2} A(A+1) = \frac{1}{6}(1 + N^{-1}). \end{aligned}$$

This completes the proof.

The preceding computation shows that, although $\Pr(x_n > x_{n+1} > x_{n+2}) \neq 1/6$, this number is approached as $N \to \infty$. In this sense, Multiply sequences with $\theta = 0$ are *asymptotically* equipartitioned by 3's. We shall show much more: Multiply sequences with any θ are *asymptotically completely equidistributed* as $N \to \infty$.

THEOREM 6. *Let θ and x_0 be fixed. For each $N = 2, 3, \cdots$ form the sequence $x_n = x_n(N)$ from the formula*

$$(3.17)\qquad x_{n+1} = \{Nx_n + \theta\} \qquad (n = 0, 1, 2, \cdots).$$

For almost all x_0 these sequences are all equidistributed, and

$$(3.18)\qquad \lim_{N\to\infty} \Pr(a_r \leqq x_{n+r}(N) < b_r \quad \text{for} \quad r = 0, 1, \cdots, k-1) = \prod_{r=0}^{k-1} (b_r - a_r)$$

if $0 \leqq a_r < b_r \leqq 1$ $(r = 0, \cdots, k-1)$. This result holds for all positive integers k.

Proof. This result follows from a calculation with Fourier series. For all real x define the periodic function $\phi(x; a, b) = \phi(x + 1; a, b)$ such that

$$(3.19)\qquad \phi = 1 (a \leqq x < b); \qquad \phi = 0 \quad (0 \leqq x < a \quad \text{or} \quad b \leqq x < 1).$$

If $0 \leqq z < b \leqq 1$ this function is discontinuous unless $a = 0$ and $b = 1$. For all sufficiently small $\epsilon > 0$ we define continuous, periodic, piecewise linear functions $\phi^+(x; a, b)$ and $\phi^-(x; a, b)$ as follows: If $a = 0$ and $b = 1$, define $\phi^+ = \phi^- = \phi \equiv 1$. If $a > 0$ or $b < 1$ define

$$(3.20)\qquad \begin{aligned} &\phi^+ = \epsilon^{-1}(x - (a - \epsilon)) && (a - \epsilon \leqq x \leqq a) \\ &\phi^+ = 1 && (a \leqq x \leqq b) \\ &\phi^+ = \epsilon^{-1}(b + \epsilon - x) && (b \leqq x \leqq b + \epsilon) \\ &\phi^+ = 0 && (b + \epsilon \leqq x \leqq a - \epsilon + 1) \\ &\phi^+(x; a, b) \equiv \phi^+(x + 1; a, b) && \text{for all} \quad x. \end{aligned}$$

Similarly define

$$\begin{aligned}
&\phi^- = \epsilon^{-1}(x - a) && (a \leqq x \leqq a + \epsilon)\\
&\phi^- = 1 && (a + \epsilon \leqq x \leqq b - \epsilon)\\
&\phi^- = \epsilon^{-1}(b - x) && (b - \epsilon \leqq x \leqq b) \qquad (3.21)\\
&\phi^- = 0 && (b \leqq x \leqq a + 1)\\
&\phi^-(x; a, b) \equiv \phi^-(x + 1; a, b) && \text{for all } x.
\end{aligned}$$

Both functions ϕ^+ and ϕ^- have uniformly and absolutely convergent Fourier series

$$\phi^{\pm}(x; a, b) = \sum_{\nu} c_\nu^{\pm}(a, b) \exp 2\pi i\nu x \quad \text{for all} \quad x. \tag{3.22}$$

Furthermore, for all x

$$\phi^-(x; a, b) \leqq \phi(x; a, b) \leqq \phi^+(x; a, b) \tag{3.23}$$

and

$$\int_0^1 \phi^{\pm}(x; a, b)\, dx = c_0^{\pm}(a, b) = b - a \pm \epsilon' \tag{3.24}$$

where $\epsilon' = \epsilon$ unless $a = 0$ and $b = 1$, in which case $\epsilon' = 0$.

For each integer $N > 1$ the sequence x_n is equidistributed for almost all x_0. Therefore, since the set of N's is denumerable, all sequences $x_n^{(N)}$ are equidistributed for almost all x_0; in the rest of the proof we assume x_0 to have any value such that all the sequences $x_n^{(N)}$ are equidistributed.

A trivial inductive proof shows that for each $n = 0, 1, \cdots$

$$x_{n+r} \equiv N^r x_n + \theta_r (\text{mod } 1) \qquad (r = 0, 1, \cdots) \tag{3.25}$$

where $\theta_r = (N^r - 1)\,\theta/(N - 1)$. Therefore, $a_r \leqq x_{n+r} < b_r$ for $r = 0, \cdots, k - 1$ if and only if

$$\prod_{r=0}^{k-1} \phi(N^r x_n + \theta_r ; a_r, b_r) = 1. \tag{3.26}$$

By the equidistribution of x_n

$$\begin{aligned}
&\Pr(a_r \leqq x_{n+r} < b_r \quad \text{for} \quad r = 0, 1, \cdots, k - 1)\\
&\quad = \lim_{M\to\infty} M^{-1} \sum_{n=1}^{M} \prod_{r=0}^{k-1} \phi(N^r x_n + \theta_r ; a_r, b_r)\\
&\quad = \int_0^1 \prod_{r=0}^{k-1} \phi(N^r x + \theta_r ; a_r, b_r)\, dx .
\end{aligned} \tag{3.27}$$

Denoting this probability by P, we find from the inequalities (3.23)

$$\int_0^1 \prod_{r=0}^{k-1} \phi^-(N^r x + \theta_r ; a_r, b_r)\, dx \leqq P \leqq \int_0^1 \prod_{r=0}^{k-1} \phi^+(N^r x + \theta_r ; a_r, b_r)\, dx. \tag{3.28}$$

Using the Fourier series (3.22) for ϕ^+, we find

$$P \leqq \int_0^1 \prod_{r=0}^{k-1} \sum_{\nu_r} c_{\nu_r}^{+}(a_r, b_r) \exp 2\pi i\nu_r(N^r x + \theta_r)\, dx. \tag{3.29}$$

Term-by-term integration gives

$$P \leqq \sum_{\nu_0} \cdots \sum_{\nu_{k-1}} c_{\nu_0}^{+}(a_0, b_0) \cdots c_{\nu_{k-1}}^{+}(a_{k-1}, b_{k-1}) I_\nu(N) \tag{3.30}$$

where

$$I_\nu(N) = \omega_\nu \int_0^1 \exp\,(2\pi i(\nu_0 + \nu_1 N + \cdots + \nu_{k-1}N^{k-1})x)\,dx \tag{3.31}$$

where $\omega_\nu = \exp 2\pi i \sum \nu_r \theta_r$. Unless all $\nu_r = 0$ the integer $\nu_0 + \nu_1 N + \cdots + \nu_{k-1}N^{k-1}$ is non-zero for all sufficiently large N, and $I_\nu(N) = 0$. Thus $I_\nu(N) \to 0$ as $N \to \infty$ for each non-zero lattice point $\nu = (\nu_0, \cdots, \nu_{k-1})$. Since all the Fourier series for the functions $\phi^+(N^r x + \theta_r\,; a_r, b_r)$ converge uniformly with respect to N and x, we may take the limit of the sum in (3.30) as $N \to \infty$ term-by-term to obtain

$$\limsup_{N\to\infty} P \leqq c_0^{+}(a_0, b_0) \cdots c_0^{+}(a_{k-1}, b_{k-1}).$$

From the identity (3.24) we conclude

$$\limsup_{N\to\infty} P \leqq \prod_{r=0}^{k-1} (b_r - a_r + \epsilon_r').$$

Working with ϕ^- instead of ϕ^+, we find similarly

$$\liminf_{N\to\infty} P \geqq \prod_{r=0}^{k-1} (b_r - a_r - \epsilon_r').$$

Since $\epsilon_r' \leqq \epsilon$ is arbitrarily small, we obtain the required result $P \to \prod (b_r - a_r)$ as $N \to \infty$.

From this result we can show that for almost all starting values x_0 the Multiply sequences $x_n(N)$ are *asymptotically completely equipartitioned* as $N \to \infty$.

THEOREM 7. *For almost all x_0 the Multiply sequences $x_n(N)$ defined by* (3.17) *have the property*

$$\lim_{N\to\infty} \Pr(z^{(n)}(N) \quad \text{in} \quad O_j) = 1/k! \tag{3.32}$$

where O_j is any of the $k!$ partitions (3.5) *of C_k, and where $z^{(n)}(N) = (x_n(N), \cdots, x_{n+k-1}(N))$.*

Proof. Let $f(z_1, \cdots, z_k) = 1$ if z lies in O_j, $f(z) = 0$ otherwise. By the equidistribution of each sequence $x_n(N)$, we have the existence of the limit

$$\begin{aligned} &\int_0^1 f(x, \{Nx + \theta_1\}, \cdots, \{N^{k-1}x + \theta_{k-1}\})\,dx \\ &\quad = \lim_{M\to\infty} \frac{1}{M} \sum_{n=1}^{M} f(x_n, \{Nx_n + \theta_1\}, \cdots, \{N^{k-1}x_n + \theta_{k-1}\})\,dx \\ &\quad = \lim_{M\to\infty} \frac{1}{M} \sum_{n=1}^{M} f(z^{(n)}(N)) = \Pr\,(z^{(n)}(N) \quad \text{in} \quad O_j) \end{aligned} \tag{3.33}$$

where the numbers θ_r are defined in (3.25). Let $\epsilon > 0$ be given. Since $f(z)$ is Riemann-integrable in C_k, we can partition C_k into k-dimensional boxes B_ν with volumes ΔV_ν such that

$$\sum_\nu M_\nu \Delta V_\nu - \epsilon \leqq \int \cdots \int_{C_k} f(z)\,dz_1 \cdots dz_k \leqq \sum_\nu m_\nu \Delta V_\nu + \epsilon \tag{3.34}$$

where $m_\nu \leqq f(z) \leqq M_\nu$ for z in B_ν. Let $P_\nu(N; M)$ be $1/M$ times the number of points $z^{(1)}(N), \cdots, z^{(M)}(N)$ which lie in B_ν. Then

$$\sum_\nu m_\nu P_\nu(N; M) \leqq \frac{1}{M} \sum_{n=1}^{M} f(z^{(n)}(N)) \leqq \sum_\nu M_\nu P_\nu(N; M). \tag{3.35}$$

By the last theorem we know that $P_\nu(N; M)$ tends to a limit $P_\nu(N)$ as $M \to \infty$ (the form of this limit is given by the integral in (3.27)) and that $P_\nu(N)$ tends to the limit ΔV_ν as $N \to \infty$. Therefore, by (3.35)

$$\sum_\nu m_\nu P_\nu(N) \leqq \Pr(z^{(n)}(N) \text{ in } O_j) \leqq \sum_\nu M_\nu P_\nu(N) \tag{3.36}$$

and

$$\begin{aligned} \limsup_{N\to\infty} \Pr(z^{(n)}(N) \text{ in } O_j) &\leqq \sum_\nu M_\nu \Delta V_\nu \\ \liminf_{N\to\infty} \Pr(z^{(n)}(N) \text{ in } O_j) &\geqq \sum_\nu m_\nu \Delta V_\nu . \end{aligned} \tag{3.37}$$

Using (3.34) and letting $\epsilon \to 0$ we obtain the existence of the limit

$$\begin{aligned} \lim_{N\to\infty} \Pr\,(z^{(n)}(N) \text{ in } O_j) &= \int_{C_k} f(z)\, dz \\ &= \text{volume of } O_j = 1/k! \end{aligned} \tag{3.38}$$

This completes the proof.

We have, in fact, proved the more general result:

THEOREM 8. *Let $f(x_1, \cdots, x_k)$ be any Riemann-integrable function in C_k for which all the one-dimensional Riemann integrals (3.33) over line segments in C_k exist. Then for almost all starting values x_0 the Multiply sequences $x_n(N)$ defined by (3.17) have the limiting property*

$$\begin{aligned} &\lim_{N\to\infty} \lim_{M\to\infty} \frac{1}{M} \sum_{n=1}^{M} f(x_n(N), \cdots, x_{n+k-1}(N)) \\ &\qquad = \int_0^1 \cdots \int_0^1 f(z_1, \cdots, z_k)\, dz_1 \cdots dz_k . \end{aligned} \tag{3.39}$$

Next we compute the autocorrelation function (1.9) of a Multiply sequence.

THEOREM 9. *For almost all x_0 the sequence defined by $x_{n+1} = Nx + \theta$, $N =$ integer > 1, has autocorrelation function*

$$R(\tau) = N^{-\tau}(\tfrac{1}{12} - \tfrac{1}{2}\{\beta\} + \tfrac{1}{2}\{\beta\}^2) \qquad (\tau = 0, 1, \cdots) \tag{3.40}$$

where $\beta = (N^\tau - 1)\theta/(N - 1)$. Thus

$$-\tfrac{1}{24}N^{-\tau} \leqq R(\tau) \leqq \tfrac{1}{12}N^{-\tau} \qquad (\tau = 0, 1, \cdots)$$

so that $R(\tau) \to 0$ as $N \to \infty$ uniformly in τ for $\tau \neq 0$.

Proof. We use Jagerman's Theorem (1.9), (1.10). We have

$$F(\tau, k, \nu) = \lim_{M\to\infty} \frac{1}{2M} \sum_{j-1}^{M} (\cos 2\pi z_j^- - \cos 2\pi z_j^+) \tag{3.41}$$

where $z_j^{\pm} = kx_j \pm x_{j+\tau}$. Letting "$\equiv$" mean "congruent modulo 1", we find from (3.25)

$$\begin{aligned} z_j^{\pm} &\equiv k\left(N^j x_0 + \frac{N^j - 1}{N - 1}\,\theta\right) \pm \nu\left(N^{j+\tau} x_0 + \frac{N^{j+\tau} - 1}{N - 1}\,\theta\right) \\ &\equiv N^j x_0' + \frac{N^j - 1}{N - 1}\,\theta' + \nu\beta \end{aligned} \tag{3.42}$$

where

$$x_0' = (k \pm \nu N^\tau) x_0, \qquad \theta' = (k \pm \nu N)\theta, \qquad \beta = \pm \frac{N^\tau - 1}{N - 1}\,\theta. \tag{3.43}$$

From (3.42) we observe, again using (3.25), that $z_j^{\pm} \equiv x_j' + \nu\beta$, where x_j' is the Multiply sequence satisfying

$$x_{j+1}' \equiv \{N x_j' + \theta'\} \qquad (j = 0, 1, \cdots). \tag{3.44}$$

The numbers x_j' are equidistributed for almost all x_0', hence by (3.43) equidistributed for almost all x_0 if $k \pm \nu N^\tau \neq 0$. By Weyl's criterion the translates $z_j \equiv x_j' + \nu\beta$ are equidistributed when the x_j' are equidistributed. Therefore,

$$\lim_{M\to\infty} \frac{1}{M} \sum_{j=1}^{M} \cos 2\pi z_j^{\pm} = \int_0^1 \cos 2\pi z \, dz = 0 \tag{3.45}$$

for almost all x_0 if the integer $k \pm \nu N^\tau \neq 0$. But $k + \nu N^\tau \geqq 2$ for all positive k and ν. Therefore, by (3.41),

$$\begin{aligned} F(\tau, k, \nu) &= 0 \quad \text{unless} \quad k - \nu N^\tau = 0 \\ F(\tau, \nu N^\tau, \nu) &= \tfrac{1}{2} \cos 2\pi\nu\beta. \end{aligned} \tag{3.46}$$

The last identity follows from $z_j^- = \nu\beta$ for all j if $k = \nu N^\tau$. By Jagerman's theorem (1.11) and by (3.46)

$$\begin{aligned} R(\tau) &= \sum_{k=1}^{\infty} \sum_{\nu=1}^{\infty} \frac{F(\tau, k, \nu)}{\pi^2 k \nu} \\ &= \sum_{\nu=1}^{\infty} \frac{\cos 2\pi\nu\beta}{2\pi^2 \nu^2 N^\tau}. \end{aligned} \tag{3.47}$$

Using the well-known identity

$$\sum_{\nu=1}^{\infty} \frac{\cos 2\pi\nu x}{\pi^2 \nu^2} = \frac{1}{6} - x + x^2 \qquad (0 \leqq x \leqq 1)$$

we obtain the required result (3.40).

We should like also to compute the spectral density

$$\begin{aligned} \phi(\omega) &= R(0) + 2 \sum_{\tau=1}^{\infty} R(\tau) \cos 2\pi\tau\omega \\ &= \frac{1}{12} + \sum_{\tau=1}^{\infty} N^{-\tau} \left(\frac{1}{6} - \left\{ \frac{N^\tau - 1}{N - 1}\,\theta \right\} + \left\{ \frac{N^\tau - 1}{N - 1}\,\theta \right\}^2 \right) \cos 2\pi\tau\omega. \end{aligned} \tag{3.48}$$

If $\theta = 0$, i.e., if $x_n = \{N^n x_0\}$, this series may be summed to the value

$$
\begin{aligned}
\phi(\omega) &= \frac{1}{12} + \sum_{\tau=1}^{\infty} N^{-\tau}\left(\frac{1}{6}\right)\cos 2\pi\tau\omega \\
&= -\frac{1}{12} + \frac{1}{6}\operatorname{Re}(1 - N^{-1}e^{2\pi i\tau\omega})^{-1} \qquad (3.49)\\
\phi(\omega) &= \frac{1}{12}\,\frac{1 - N^{-2}}{1 - 2N^{-1}\cos 2\pi\omega + N^{-2}}.
\end{aligned}
$$

Multiply sequences and Weyl sequences are generated by recurrence formulas $x_{n+1} = \{p(x_n)\}$, where $p(x)$ is a polynomial of first degree. In the next theorem we assert that there are no polynomials of higher degree which generate equidistributed sequences.

Theorem 10. *Let $p(x)$ be a polynomial with real coefficients. Suppose that for some x_0 the sequence $x_1, x_2, \cdots$ generated by*

$$x_{n+1} = \{p(x_n)\} \qquad (n = 0, 1, \cdots) \tag{3.50}$$

is equidistributed in $0 \leqq x < 1$. Then either $p(x) = x + \alpha$, α irrational, or $p(x) = Nx + \theta$, $\pm N =$ integer > 1.

Proof. For any assertion S_n we have $\Pr(S_n) = \Pr(S_{n+1})$, since the number of numbers $n = 1, \cdots, M$ for which S_n is true differs by at most 1 from the corresponding number for S_{n+1}. In particular,

$$
\begin{aligned}
\Pr(x_n < y) &= \Pr(x_{n+1} < y) \\
&= \Pr(\{p(x_n)\} < y)
\end{aligned} \tag{3.51}
$$

or, by equidistribution, if $0 < y < 1$,

$$y = m(x|\{p(x)\} < y, \qquad 0 < x < 1), \tag{3.52}$$

i.e., y is the measure of the finite collection of intervals of values x such that $\{p(x)\} < y$, $0 < x < 1$. There is at most a finite number of points z_i such that $p'(z_i) = 0$. Let $y = b$ be any point satisfying

$$0 < b < 1, \qquad b \neq \{p(0)\}, \qquad b \neq \{p(1)\}, \qquad b \neq \{p(z_i)\} \quad \text{for any } z_i. \tag{3.53}$$

Let $x = a_1, a_2, \cdots, a_k$ be the finite collection of points in the open interval $(0, 1)$ such that $b = \{p(a_j)\}$. There must be at least one such point a_1, for otherwise there would be an interval of values surrounding b which were not achieved by $p(x)$; then none of the values $x_1, x_2, \cdots$ could be in this interval, and the sequence x_n could not be equidistributed. Let

$$N_j = [p(a_j)], \qquad b = p(a_j) - N_j \qquad (j = 1, \cdots, k). \tag{3.54}$$

Since $p'(a_j) \neq 0$, there are inverse functions $\xi_j(y)$ uniquely defined in a neighborhood $|y - b| < \delta$ such that

$$
\begin{aligned}
&\xi_j(b) = a_j, \qquad y = \{p(\xi_j(y)\} = p(\xi_j(y)) - N_j, \\
&p'(x_j) \neq 0, \qquad 0 < \xi_j(y) < 1 \qquad (j = 1, \cdots, k).
\end{aligned} \tag{3.55}
$$

For $|y - b| < \delta$ differentiation of the identity (3.52) gives

$$1 = \frac{d}{dy} m(x \,|\{p(x)\} < y, \quad 0 < x < 1) \tag{3.56}$$
$$= \sum_{j=1}^{k} \left| \frac{d\xi_j}{\mathrm{dy}} \right| = \sum_{j=1}^{k} \theta_j \frac{d\xi_j(y)}{dy}$$

where $\theta_j = \operatorname{sgn} \xi'(b) = \operatorname{sgn} p'(a_j) = \pm 1$.

We now consider each function $\xi_j(y)$ in the whole complex y-plane. This function is analytic except for branch points $y_{j\nu} = p(z_\nu) - N_j$ where $z_1, z_2, \cdots$ are the zeros of $p'(z)$. For $j = 1, \cdots, k$ there are at most $k(d - 1)$ distinct points $y_{j\nu}$ if d is the degree of $p(z)$. Let the y-plane be cut by non-intersecting curves extending from the points $y_{j\nu}$ to the point at infinity. In the cut plane every function $\xi_j(y)$ is a uniquely-defined analytic function of y satisfying

$$y = p(\xi_j(y)) - N_j \qquad (j = 1, \cdots, k). \tag{3.57}$$

Analytic continuation of (3.56) gives

$$1 = \sum_{j=1}^{k} \theta_j \frac{d\xi_j(y)}{dy} = \sum_{j=1}^{k} \frac{\theta_j}{p'(\xi_j(y))} \tag{3.58}$$

in the whole cut plane. Let y tend to infinity in the cut plane. By (3.57) each point $\xi_j(y)$ tends to infinity because d = degree of $p \geqq 1$. If $d > 1$, then (3.58) gives a contradiction, since $\xi_j(y) \to \infty$ would imply $p'(\xi_j(y)) \to \infty$ and $1 = 0$ in the limit.

Therefore $d = 1$, and $p(x) = Ax + B$, $A \neq 0$. Then $\theta_j = \operatorname{sgn} \xi_j'(b) = \operatorname{sgn} A$. Equation (3.58) now yields $1 = k/A$, $A = \pm$ the positive integer k. The case $A = -1$ is impossible because $x_{n+1} = \{-x_n + B\}$ gives the two-valued sequence

$$x_{2m} = \{x_0\}, \qquad x_{2m+1} = \{-x_0 + B\} \qquad (m = 0, 1, \cdots).$$

If $A = 1$ we must have B equal to an irrational number α; otherwise x_n is periodic. If $k = 2, 3, \cdots$, the proof is completed by setting $N = A = \pm k$, $\theta = B$.

With regard to sequences generated by $x_{n+1} = \{Nx_n + \theta\}$ where N = integer < -1, the argument in [7] proving that x_n is equidistributed for almost all x_0 was made only for N = integer > 1; but the proof also holds for N = integer < -1.

4. Polynomial Sequences. Weyl [1] proved that for any integer $p > 0$ the sequence

$$x_n = \{\alpha n^p + c_1 n^{p-1} + c_2 n^{p-2} + \cdots + c_p\} \qquad (n = 0, 1, \cdots) \tag{4.1}$$

is equidistributed in C_1 if α is irrational. We shall study some sequential properties of these "polynomial" sequences.

THEOREM 11. *If the leading coefficient α is irrational, a polynomial sequence* (4.1) *of degree p is equidistributed by p's but is not equidistributed by $(p + 1)$'s.*

Remark. A sequence x_n equidistributed by r's is equidistributed by j's for all $j < r$. This follows directly from the definition (1.2) of equidistribution if we set $z_i^{(n)} = x_{r+i-1}$ $(i = 1, \cdots, r)$, $a_{j+1} = \cdots = a_r = 0$, $b_{j+1} = \cdots = b_r = 1$. Therefore, Theorem 11 implies that the polynomial sequence (4.1) is equidistributed by k's if and only if $k \leqq p$.

Proof. First we show that x_n is not equidistributed by $(p+1)$'s. Let $f(n) = \alpha n^p + \cdots + c_p$. Let Δ be the forward-difference operator: $\Delta f(n) = f(n+1) - f(n)$. Then

$$\Delta^p f(n) = p!\alpha$$

or

$$k_0 f(n) + k_1 f(n+1) + \cdots + k_p f(n+p) = p!\alpha$$

where k_ν is the constant

$$k_\nu = (-1)^{p-\nu}\binom{p}{\nu} \qquad (\nu = 0, 1, \cdots, p).$$

For this choice of the integers k_ν we have, for all N,

$$\frac{1}{N}\sum_{n=0}^{N-1} \exp 2\pi i(k_0\, x_n + k_1\, x_{n+1} + \cdots + k_p\, x_{n+p})$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} \exp 2\pi i(k_0\, f(n) + \cdots + k_p\, f(n+p)) = \exp 2\pi i p!\alpha \neq 0.$$

Therefore, by the Weyl criterion, the vectors

$$Z^{(n)} = (x_n, x_{n+1}, \cdots, x_{n+p})$$

are not equidistributed in C_{p+1}, i.e., the sequence x_n is not equidistributed by $(p+1)$'s.

However, if $k_0, \cdots, k_{p-1}$ are any p integers not all zero, we have

$$\sum_{\nu=0}^{p-1} k_\nu f(n+\nu) = \sum_{\nu=0}^{p-1} k_\nu' \Delta^\nu f(n) \tag{4.2}$$

where

$$k_\nu' = \binom{\nu}{\nu} k_\nu + \binom{\nu+1}{\nu} k_{\nu+1} + \cdots + \binom{p-1}{\nu} k_{p-1}. \tag{4.3}$$

The integers k_ν' are not all zero; in fact, $k_s' = k_s$ if $\nu = s$ is the largest integer $\leqq p-1$ such that $k_\nu \neq 0$. Let $\nu = r$ be the smallest integer such that $k_\nu' \neq 0$. Then the polynomial (4.2) is a polynomial of degree $p - r \geqq 1$ with irrational leading coefficient

$$\beta = p(p-1)\cdots(p-r+1)\alpha k_r'.$$

Then the polynomial (4.2) has the form $\beta n^{p-r} + \cdots$, and by the equidistribution of polynomials (mod 1) with irrational leading coefficients

$$\frac{1}{N}\sum_{n=0}^{N-1} \exp 2\pi i(\beta n^{p-r} + \cdots) \to 0 \quad \text{as} \quad N \to \infty \tag{4.4}$$

or, since $x_\nu = \{f(\nu)\}$,

$$\frac{1}{N}\sum_{n=0}^{N-1} \exp 2\pi i(k_0\, x_n + k_1\, x_{n+1} + \cdots + k_{n+p-1}\, x_{n+p-1}) \to 0. \tag{4.5}$$

Therefore, the vectors $z^{(n)} = (x_n, \cdots, x_{n+p-1})$ are equidistributed in C_p, i.e., the sequence x_n is equidistributed by p's.

By using the identity (1.11), Jagerman [5] showed that the sequence $x_n = \{n^2\alpha\}$ is white. Without using Jagerman's identity, one can derive in an elementary way the general result:

THEOREM 12. *If* $p \geqq 2$ *every sequence* $x_n = \{\alpha n^p + c_1 n^{p-1} + \cdots + c_p\}$, α *irrational, is white.*

Proof. Let τ be any positive integer. For every integer k_0, k_1 not both zero, we have, as in the preceding proof,

$$k_0(\alpha n^p + c_1 n^{p-1} + \cdots) + k_1(\alpha(n+\tau)^p + c_1(n+\tau)^{p-1} + \cdots) = \text{polynomial in } n \text{ of degree} \geqq 1 \text{ with irrational leading coefficient.}$$

Therefore, the vectors $(x_n, x_{n+\tau})$ $\quad (n = 0, 1, \cdots)$ are equidistributed in C_2. Therefore,

$$\begin{aligned} R(\tau) &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \left(x_n - \frac{1}{2}\right)\left(x_{n+\tau} - \frac{1}{2}\right) \\ &= \int_0^1 \int_0^1 \left(x - \frac{1}{2}\right)\left(y - \frac{1}{2}\right) dx\, dy = 0. \end{aligned}$$

We know from Theorem 11 that no choice of α will make the sequence $x_n = \{n^2\alpha\}$ equidistributed by threes. Can we nevertheless choose α so as to make this sequence equipartioned by threes?

THEOREM 13. *The sequence* $x_n = \{n^2\alpha\}$, α *irrational,* $0 < \alpha < 1$, *is equipartitioned by threes if and only if* α *is one of the four numbers* $(3 \pm \sqrt{3})/12$, $(9 \pm \sqrt{3})/12$.

Proof. First we shall find values α which make

$$\Pr(x_n > x_{n+1} > x_{n+2}) = \tfrac{1}{6}. \tag{4.6}$$

An acceptable value α must also make the five other orderings of x_n, x_{n+1}, x_{n+2} occur with probability $1/6$. In the unit cube C_3 we define the characteristic function $\phi(x, y, z)$ of the set $x > y > z$: $\phi = 1$ for points in the set, $\phi = 0$ for other points in C_3. The relation (4.6) is equivalent to the existence of the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \phi(x_n, x_{n+1}, x_{n+2}) = \frac{1}{6}. \tag{4.7}$$

The function $\phi(x, y, z)$ has a Fourier series:

$$\phi(x, y, z) \sim \sum_{p,q,r} c_{pqr} \exp 2\pi i(px + qy + rz). \tag{4.8}$$

If ϕ is defined by periodicity outside C_3, this series converges to the piecewise constant function ϕ except at the points of discontinuity. In particular, since the numbers $x_\nu = \{\nu^2\alpha\}$, α irrational, are distinct positive numbers < 1, we have

$$\begin{aligned} \phi(x_n, x_{n+1}, x_{n+2}) &= \sum_{p,q,r} c_{pqr} \exp 2\pi i(px_n + qx_{n+1} + rx_{n+2}) \\ &= \sum_{p,q,r} c_{pqr} \exp 2\pi i(pn^2 + q(n+1)^2 + r(n+2)^2)\alpha. \end{aligned} \tag{4.9}$$

Observing that $pn^2 + q(n + 1)^2 + r(n + 2)^2$ is independent of n if and only if p, q, r are proportional to 1, -2, 1, we compute

$$(4.10)\quad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \exp(px_n + qx_{n+1} + rx_{n+2}) = \begin{cases} e^{4\pi i r\alpha} & \text{if } p = r, q = -2r \\ 0 & \text{otherwise.} \end{cases}$$

If we can justify the interchange of limits

$$(4.11)\quad \begin{aligned} &\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{p,q,r} c_{p,q,r} \exp 2\pi i(px_n + qx_{n+1} + rx_{n+2}) \\ &\qquad = \sum_{p,q,r} c_{pqr} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \exp 2\pi i(px_n + qx_{n+1} + rx_{n+2}) \end{aligned}$$

we shall find, by (4.10),

$$(4.12)\quad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \phi(x_n , x_{n+1} , x_{n+2}) = \sum_{r=-\infty}^{\infty} c_r\, e^{4\pi i r\alpha}$$

where $c_r = c_{r,-2r,r}$. We compute the Fourier coefficient

$$(4.13)\quad \begin{aligned} c_r &= \int_0^1\int_0^1\int_0^1 \phi(x, y, z) e^{-2\pi i r(x-2y+z)}\, dx\, dy\, dz \\ c_r &= \int_0^1\int_0^x\int_0^y e^{-2\pi i r(x-2y+z)}\, dz\, dy\, dx \\ c_r &= \frac{1}{4\pi^2 r^2}\,(r \neq 0),\, c_0 = \frac{1}{6}. \end{aligned}$$

Then formula (4.12) gives

$$(4.14)\quad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \phi(x_n , x_{n+1} , x_{n+2}) = \frac{1}{6} + \sum_{r=1}^{\infty} \frac{\cos 4\pi r\alpha}{2\pi^2 r^2}.$$

The series on the right has the sum

$$(4.15)\quad \begin{aligned} S(\alpha) &= \tfrac{1}{4} - \alpha + 2\alpha^2 \qquad (0 \leqq \alpha \leqq \tfrac{1}{2}) \\ S(\alpha) &= S(\alpha - \tfrac{1}{2}) \qquad (\tfrac{1}{2} \leqq \alpha \leqq 1). \end{aligned}$$

We require those values of α which make $S(\alpha) = 1/6$. Solving the quadratic equation, we find

$$(4.16)\quad \alpha = (3 \pm \sqrt{3})/12 \quad (0 < \alpha < \tfrac{1}{2}), \qquad \alpha = (9 \pm \sqrt{3})/12 \quad (\tfrac{1}{2} < \alpha < 1).$$

If the interchange of limits (4.11) is justified, we have shown that $\Pr(x_n > x_{n+1} > x_{n+2}) = 1/6$ if and only if α is one of the values (4.16).

Now we shall justify the interchange of limits. The Fourier series (4.8) has partial sums

$$(4.17)\quad s_{\underline{m}}(x, y, z) = s_{m_1,m_2,m_3}(x, y, z) = \sum_{|p|\leqq m_1} \sum_{|q|\leqq m_2} \sum_{|r|\leqq m_3} c_{pqr} e^{2\pi i(px+qy+rz)}.$$

From these sums we form the Fejér means

$$(4.18)\quad \sigma_{\underline{m}}(x, y, z) = \frac{1}{(m_1 + 1)(m_2 + 1)(m_3 + 1)} \sum_{k_1=0}^{m_1} \sum_{k_2=0}^{m_2} \sum_{k_3=0}^{m_3} s_{k_1,k_2,k_3}(x, y, z).$$

Since $0 \leqq \phi \leqq 1$, we have $0 \leqq \sigma_{\underline{m}} \leqq 1$. If F is any closed subset of points of continuity in C_3, then $\sigma_{\underline{m}} \to \phi$ uniformly in F as $|\underline{m}| \to \infty$ (c.f., A. Zygmund [8]). Let $W(p, q, r)$ be the limit computed in (4.10). Since each partial sum is a finite sum, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} s_{\underline{m}}(x_n, x_{n+1}, x_{n+2}) = \sum_{|p|\leqq m_1} \sum_{|q|\leqq m_2} \sum_{|r|\leqq m_3} c_{pqr} W_{(p,q,r)} = \frac{1}{6} + \sum_{r=1}^{t} \frac{\cos 4\pi r\alpha}{2\pi r^2} \tag{4.19}$$

where t is the largest integer for which $t \leqq m_1$, $2t \leqq m_2$, $t \leqq m_3$. We note that $t = t(\underline{m}) \to \infty$ as $|\underline{m}| \to \infty$. Let the limit (4.19) be called $s(\underline{m})$. As $|\underline{m}| \to \infty$, $s(\underline{m})$ tends to the limit

$$s = \lim_{|\underline{m}|\to\infty} s(\underline{m}) = \frac{1}{6} + \sum_{r=1}^{\infty} \frac{\cos 4\pi r\alpha}{2\pi r^2}. \tag{4.20}$$

Replacing $s_{\underline{m}}$ by $\sigma_{\underline{m}}$ in (4.19), we find

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sigma_{\underline{m}}(x_n, x_{n+1}, x_{n+2}) = \sigma(\underline{m}) \tag{4.21}$$

where $\sigma(\underline{m})$ is the $(C, 1)$ mean value of the numbers $s(\underline{k})$ for $k_1 \leqq m_1$, $k_2 \leqq m_2$, $k_3 \leqq m_3$. By the regularity of $(C, 1)$ summability, we have

$$\sigma(\underline{m}) \to s = \frac{1}{6} + \sum_{r=1}^{\infty} \frac{\cos 4\pi r\alpha}{2\pi r^2} \text{ as } |\,\underline{m}\,| \to \infty. \tag{4.22}$$

Let $\epsilon > 0$ be given. Define F to be the closed set of points (x, y, z) satisfying all the inequalities

$$\begin{gathered} \epsilon \leqq x \leqq 1 - \epsilon, \qquad \epsilon \leqq y \leqq 1 - \epsilon, \qquad \epsilon \leqq z \leqq 1 - \epsilon, \\ |x - y| \geqq \epsilon, \qquad |y - z| \geqq \epsilon. \end{gathered} \tag{4.23}$$

Since ϕ is continuous in F, there is a number $m_0 = m_0(\epsilon)$ so large that

$$|\,\phi(x, y, z) - \sigma_{\underline{m}}(x, y, z)\,| < \epsilon \quad \text{in} \quad F \quad \text{if} \quad |\underline{m}| > m_0. \tag{4.24}$$

Let $R = C_3 - F$. Let $P_n = (x_n, x_{n+1}, x_{n+2})$. We have

$$\begin{aligned} \sum_{n=1}^{N} \phi(P_n) - \sum_{n=1}^{N} \sigma_{\underline{m}}(P_n) &= \sum_{\substack{n=1 \\ P_n \in F}}^{N} (\phi(P_n) - \sigma_{\underline{m}}(P_n)) + \sum_{\substack{n=1 \\ P_n \in R}}^{N} (\phi(P_n) - \sigma_{\underline{m}}(P_n)) \\ &= \textstyle\sum_1 + \sum_2. \end{aligned} \tag{4.25}$$

By (4.24) we have

$$|\textstyle\sum_1| < N\epsilon \quad \text{if} \quad |\underline{m}| > m_0(\epsilon). \tag{4.26}$$

Since ϕ and $\sigma_{\underline{m}}$ both lie between 0 and 1, we have $|\sum_2| \leqq \nu_N =$ the number of points $P_1, \cdots, P_N$ which lie in R. By the definition (4.23) every point P_n in R satisfies at least one of the inequalities

$$\begin{gathered} 0 \leqq x_n < \epsilon, \qquad 1 - \epsilon < x_n < 1, \qquad 0 \leqq x_{n+1} < \epsilon, \qquad 1 - \epsilon < x_{n+1} < 1, \\ 0 \leqq x_{n+2} < \epsilon, \qquad 1 - \epsilon < x_{n+2} < 1, \qquad |x_n - x_{n+1}| < \epsilon, \qquad |x_{n+1} - x_{n+2}| < \epsilon. \end{gathered} \tag{4.27}$$

But the sequence $x_k = \{k^2\alpha\}$ is equidistributed in C_1, and the sequence (x_k, x_{k+1}) is equidistributed in C_2. Therefore,

$$\begin{aligned}(4.28)\qquad \limsup_{N\to\infty} \frac{1}{N}\nu_N \leqq \int_0^\epsilon dx + \int_{1-\epsilon}^1 dx + \int_0^\epsilon dy + \int_{1-\epsilon}^1 dy \\ + \int_0^\epsilon dz + \int_{1-\epsilon}^1 dz + \iint_{|x-y|<\epsilon} dx\,dy + \iint_{|y-z|<\epsilon} dy\,dz < 10\epsilon.\end{aligned}$$

Letting s be the limit (4.22), we find from (4.25)

$$(4.29)\qquad \left|\frac{1}{N}\sum_{n=1}^N \phi(P_n) - s\right| \leqq \left|\frac{1}{N}\sum_{n=1}^N \sigma_{\underline{m}}(P_n) - s\right| + \frac{1}{N}\left|\textstyle\sum_1\right| + \frac{1}{N}\left|\textstyle\sum_2\right|.$$

Let $N \to \infty$. If $|\underline{m}| > m_0(\epsilon)$ we find

$$(4.30)\qquad \limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^N \phi(P_n) - s\right| < |\sigma(\underline{m}) - s| + \epsilon + 10\epsilon.$$

But by (4.22) $|\sigma(\underline{m}) - s| < \epsilon$ if $|\underline{m}|$ is sufficiently large. Therefore, the limit superior (4.30) is $<12\epsilon$ for arbitrarily small $\epsilon > 0$. Therefore,

$$(4.31)\qquad \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N \phi(P_n) = s = \frac{1}{6} + \sum_{r=1}^\infty \frac{\cos 4\pi r\alpha}{2\pi r^2}.$$

This justifies the required formula (4.12). The essential point in the proof was the inequality (4.28), which showed that not too many of the points (x_n, x_{n+1}, x_{n+2}) fell near the discontinuities of ϕ.

To discuss each of the other five orderings we proceed exactly as before. For example, to find those values of α which make $\Pr(x_n > x_{n+2} > x_{n+1}) = 1/6$, we define the characteristic function $\phi(x, y, z)$ of the set $x > z > y$ in the unit cube C_3. This function has a Fourier series with coefficients c_{pqr}. Justifying the necessary interchange of limits, we find for this new function ϕ:

$$\begin{aligned}(4.32)\qquad \Pr(x_n > x_{n+2} > x_{n+1}) &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N \phi(x_n, x_{n+1}, x_{n+2}) \\ &= \sum_{r=-\infty}^\infty c_{r,-2r,r}e^{4\pi i r\alpha} = S(\alpha)\end{aligned}$$

where in this case

$$\begin{aligned}(4.33)\qquad c_r = c_{r,-2r,r} &= \int_0^1\int_0^x\int_0^z e^{-2\pi i r(x-2y+z)}\,dx\,dz\,dy \\ c_0 = 1/6, \qquad c_r &= -\frac{1}{8\pi^2 r^2} \quad (r \neq 0).\end{aligned}$$

This gives the sum

$$\begin{aligned}(4.34)\qquad S(\alpha) &= \tfrac{1}{8} + \tfrac{1}{2}\alpha - \alpha^2 \qquad (0 \leqq \alpha \leqq \tfrac{1}{2}) \\ S(\alpha) &= S(\alpha - \tfrac{1}{2}) \qquad (\tfrac{1}{2} \leqq \alpha \leqq 1).\end{aligned}$$

This function $S(\alpha)$ is different from the function $S(\alpha)$ computed in (4.15) for the first ordering. Nevertheless, the new equation $S(\alpha) = 1/6$ has the same roots as the old equation. These four roots in the interval $0 < \alpha < 1$ are, as before,

$(3 \pm \sqrt{3})/12$, $(9 \pm \sqrt{3})/12$; and $\Pr(x_n > x_{n+2} > x_{n+1}) = 1/6$ if and only if α has one of these values.

For the remaining four cases these are the results: To compute the three probabilities $\Pr(x_{n+1} > x_n > x_{n+2})$, $\Pr(x_{n+1} > x_{n+2} > x_n)$, $\Pr(x_{n+2} > x_n > x_{n+1})$ we compute, respectively, the Fourier coefficients $c_r = c_{r,-2r,r}$ for the characteristic functions ϕ of the sets $y > x > z$, $y > z > x$, $z > x > y$. For each of these cases computation gives the coefficients c_r recorded in (4.33) for the second case. For the last case, to compute the probability $\Pr(x_{n+2} > x_{n+1} > x_n)$ we find for the characteristic function ϕ of the set $z > y > x$ the same coefficients c_r which were computed in (4.13) for the first case. These unexpected coincidences show that the same four values of α make all of the six probabilities equal to 1/6. This completes the proof of the theorem.

5. Completely Equidistributed Sequences. J. F. Koksma [3] proved in 1934 that for almost all $\theta > 1$ the sequence $x_n = \{\theta^n\}$ is equidistributed. We shall show that these sequences, unlike the other equidistributed sequences which we have investigated, are completely equidistributed. We shall use the following preliminary result:

THEOREM: (Koksma [3], Satz 3.) *Let α and β be fixed real numbers with $\alpha < \beta$. For each natural number n let $f(n, \theta)$ be a real, continuously differentiable function of θ in $\alpha \leqq \theta \leqq \beta$; and let*

$$\frac{\partial}{\partial\theta} f(m,\theta) - \frac{\partial}{\partial\theta} f(n,\theta) \tag{5.1}$$

denote for each pair of unequal natural numbers m and n a monotone function of θ in $\alpha \leqq \theta \leqq \beta$, which everywhere in this interval has absolute value $\geqq K$, where K is positive and independent of θ, m, and n. Then $f(n, \theta)(n = 1, 2, \cdots)$ is equidistributed modulo 1 for almost all θ in $\alpha \leqq \theta \leqq \beta$.

From this theorem we derive the following:

THEOREM 14. *Let $p(\theta)$ be any twice continuously differentiable function with at most a finite number of zeros in any finite subinterval of $\theta > 1$. For $n = 1, 2, \cdots$ let $M(n) \geqq 1$; and for each pair of positive integers $N \neq n$ let*

$$|M(N) - M(n)| \geqq L \tag{5.2}$$

where L is a positive number independent of N and n. Then the function $p(\theta)\theta^{M(n)}$ $(n = 1, 2, \cdots)$ is for almost all $\theta > 1$ equidistributed modulo 1.

For $p(\theta) = 1$ this theorem was proved by Koksma [3], Satz 2.

Proof. Let $1 < \alpha < \beta$, where the closed interval $\alpha \leqq \theta \leqq \beta$ contains no zero of $p(\theta)$. The open interval between two consecutive zeros of $p(\theta)$ can be covered by a denumerable collection of closed intervals $[\alpha, \beta]$. Therefore, it will suffice to show that $p(\theta)\theta^{M(n)}$ is equidistributed modulo 1 for almost all θ in each interval $[\alpha, \beta]$. Let $g(\theta) = p(\theta)\theta^{M(n)}$. Let

$$\psi(N, n, \theta) = \frac{\partial}{\partial\theta} g(N,\theta) - \frac{\partial}{\partial\theta} g(n,\theta). \tag{5.3}$$

We shall show that there is an integer $s \geqq 0$ such that, if $N > n > s$, then ψ is a monotone function of θ in $[\alpha, \beta]$ with absolute value $\geqq K$, where K is a positive number independent of N, n, and θ. Then, by Koksma's theorem, the sequence $f(n, \theta) = g(n + s, \theta) \quad (n = 1, 2, \cdots)$ will have been shown to be equidistributed modulo 1. But this will imply that the original sequence $g(n, \theta)(n = 1, 2 \cdots)$ is equidistributed modulo 1, since the first s members of a sequence cannot affect the property of equidistribution.

We show first that $M(n) \to \infty$ as $n \to \infty$. Given any $B > 1$ we let $\nu \leqq \infty$ denote the number of integers n such that $M(n) \leqq B$. Because of the inequality (5.2), we have $(\nu - 1)L \leqq B - 1$. Therefore, ν is finite. If n_B is the greatest integer n such that $M(n) \leqq B$, then $M(n) > B$ if $n > n_B$. Therefore, $M(n) \to \infty$ as $n \to \infty$.

Let A and a denote, respectively, the larger and the smaller of the values $M(N)$ and $M(n)$. Then

$$(5.4) \qquad \begin{aligned} |\psi(N, n, \theta)| &= \left| \frac{\partial}{\partial\theta}\theta^A p(\theta) - \frac{\partial}{\partial\theta}\theta^a p(\theta) \right| \\ &= |p(\theta)(A\theta^{A-1} - a\theta^{a-1}) + p'(\theta)(\theta^A - \theta^a)|. \end{aligned}$$

For $\alpha \leqq \theta \leqq \beta$ there are numbers $\epsilon > 0$ and $D \geqq 0$ such that

$$(5.5) \qquad |p(\theta)| \geqq \epsilon, \qquad |p'(\theta)| \leqq D, \qquad |p''(\theta)| \leqq D.$$

Then

$$(5.6) \qquad \begin{aligned} |\psi(N, n, \theta)| &\geqq \frac{\epsilon}{\theta}(A\theta^A - a\theta^a) - D(\theta^A - \theta^a) \\ &\geqq \left(\frac{\epsilon a}{\theta} - D\right)\theta^a(\theta^{A-a} - 1). \end{aligned}$$

But, by the assumption (5.2), $A - a \geqq L$. Therefore, for $\alpha \leqq \theta \leqq \beta$,

$$(5.7) \qquad |\psi(N, n, \theta)| \geqq K = \left(\frac{\epsilon a}{\beta} - D\right)(\alpha^L - 1).$$

The number K is positive if $a > D\beta/\epsilon$. Since $M(x) \to \infty$ as $x \to \infty$, there is a number s so large that $M(x) > D\beta/\epsilon$ if $x > s$. Therefore,

$$(5.8) \qquad |\psi(N, n, \theta)| \geqq K > 0 \quad \text{if} \quad N > n > s.$$

It remains only to show that ψ is monotone in θ for $N > n >$ sufficiently large s. We compute

$$\begin{aligned} \left|\frac{\partial\psi(N, n, \theta)}{\partial\theta}\right| &= |p(\theta)(A(A-1)\theta^{A-2} - a(a-1)\theta^{a-2}) \\ &\quad + 2p'(\theta)(A\theta^{A-1} - a\theta^{a-1}) + p''(\theta)(\theta^A - \theta^a)| \\ &\geqq \frac{\epsilon}{\theta^2}(A(A-1)\theta^A - a(a-1)\theta^a) - \frac{2D}{\theta}(A\theta^A - a\theta^a) - D(\theta^A - \theta^a) \\ &\geqq \epsilon\theta^{-2}a((A-1)\theta^A - (a-1)\theta^a) - 2D\theta^{-1}(A\theta^A - a\theta^a) - D(\theta^A - \theta^a) \\ &= (\epsilon\theta^{-2}a - 2D\theta^{-1})(A\theta^A - a\theta^a) - (D + \epsilon\theta^{-2}a)(\theta^A - \theta^a). \end{aligned}$$

If $a > 2D\beta/\epsilon$, the last expression is

$$\geqq (\epsilon\theta^{-2}a^2 - (2D\theta^{-1} + \epsilon\theta^{-2})a - D)(\theta^A - \theta^a).$$

The last expression is positive for all θ in $[\alpha, \beta]$ if

$$\epsilon\beta^{-2}a^2 - (2D\alpha^{-1} + \epsilon\alpha^{-2})a - D > 0$$

which is true for all sufficiently large a. Therefore, if s is sufficiently large,

$$\frac{\partial\psi}{\partial\theta}(N, n, \theta) \neq 0 \quad \text{if} \quad N > n > s. \tag{5.9}$$

This gives the required monotonicity; inequalities (5.8) and (5.9) complete the proof of the theorem.

THEOREM 15. *For any two functions $A(\theta)$, $B(\theta)$ the sequence $x_n = \{A(\theta)\theta^n + B(\theta)\}$ $(n = 1, 2, \cdots)$ is completely equidistributed for almost all $\theta > 1$ if $A(\theta)$ has two continuous derivatives for $\theta > 1$ and if $A(\theta)$ has at most a finite number of zeros in any finite subinterval of $\theta > 1$.*

Proof. For every integer $r > 0$ we must show that for almost all $\theta > 1$ the sequence of vectors $z^{(n)} = (x_n, x_{n+1}, \cdots, x_{n+r-1})$ $(n = 1, 2, \cdots)$ is equidistributed in the r-dimensional unit cube C_r. By the Weyl criterion this is equivalent to showing that

$$\frac{1}{N}\sum_{n=1}^{N} \exp 2\pi i \sum_{\rho=0}^{r-1} k_\rho(A(\theta)\theta^{n+\rho} + B(\theta)) \to 0 \quad \text{as} \quad N \to \infty \tag{5.10}$$

for almost all $\theta > 1$ if $k_0, \cdots, k_{r-1}$ are any integers not all zero. Let $q(\theta) = k_0 + k_1\theta + \cdots + k_{r-1}\theta^{r-1}$. Formula (5.10) requires that

$$\frac{1}{N}\sum_{n=1}^{N} \exp 2\pi i q(\theta)A(\theta)\theta^n \to 0 \quad \text{as} \quad N \to \infty. \tag{5.11}$$

But the function $p(\theta) = q(\theta)A(\theta)$ satisfies the conditions of the preceding theorem, since the polynomial q has at least one non-zero coefficient. Therefore, for almost all $\theta > 1$ the sequence $\{q(\theta)A(\theta)\theta^n\}$ is equidistributed in C_1. Then (5.11) follows from the one-dimensional form of the Weyl criterion applied to the sequence $q(\theta)A(\theta)\theta^n$. This completes the proof.

COROLLARY. *The sequence $\{\theta^n\}$ $(n = 1, 2, \cdots)$ is completely equidistributed for almost all $\theta > 1$.*

Proof. In the last theorem this is the case $A(\theta) = 1$, $B(\theta) = 0$.

The algebraic character of the number θ may influence the sequential properties of $\{\theta^n\}$. Although it is not known whether $\{\theta^n\}$ may be equidistributed if θ is rational, we have the result:

THEOREM 16. *If $\{\theta^n\}$ is equidistributed by r's, then θ cannot be an algebraic number of degree $< r$. In particular, if $\{\theta^n\}$ is completely equidistributed, then θ is transcendental.*

Proof. By (5.11), if $\{\theta^n\}$ is equidistributed by r's, then

$$\frac{1}{N}\sum_{n=1}^{N} \exp 2\pi i(k_0 + k_1\theta + \cdots + k_{r-1}\theta^{r-1})\theta^n \to 0 \quad \text{as} \quad N \to \infty$$

if $k_0, \cdots, k_{r-1}$ are distinct integers not all zero. Therefore, $q(\theta) \neq 0$ for any polynomial q of degree $< r$ with integer coefficients not all zero.

THEOREM 17. *Every completely equidistributed sequence* x_n *is white.*

Proof. For every $\tau = 1, 2, \cdots$ the pairs $(x_n, x_{n+\tau})$ are equidistributed in C_2, since for $0 \leqq a_0 < b_0 \leqq 1$, $0 \leqq a_\tau < b_\tau \leqq 1$

$$\begin{aligned}\Pr(a_0 \leqq x_n < b_0,\ \ a_\tau \leqq x_{n+\tau} < b_\tau)\\ = \Pr(a_0 \leqq x_n < b_0,\ \ 0 \leqq x_{n+1} < 1, \cdots,\ \ 0 \leqq x_{n+\tau-1} < 1,\ \ a_\tau \leqq x_{n+\tau} < b_\tau)\\ = (b_0 - a_0)(b_\tau - a_\tau)\end{aligned}$$

because the sequence x_n is equidistributed by $(\tau + 1)$'s. Therefore, as in the proof of Theorem 12, x_n is white.

6. White Sequences. We have called an equidistributed sequence x_n white if it is uncorrelated with any of its translates $x_{n+\tau}$, $\tau \neq 0$:

$$R(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left(x_n - \frac{1}{2}\right)\left(x_{n+\tau} - \frac{1}{2}\right) = 0 \qquad (\tau = 1, 2, \cdots). \tag{6.1}$$

We have shown in the proof of Theorem 12 that every sequence x_n is white for which the pairs $(x_n, x_{n+\tau})$ are equidistributed in C_2 for every $\tau = 1, 2, \cdots$. The purpose of this section is to emphasize that whiteness is a weak criterion of randomness. We shall show that an equidistributed white sequence need not be equipartitioned by twos.

THEOREM 18. *There is an equidistributed white sequence* x_n *for which* $\Pr(x_n > x_{n+1}) > \frac{1}{2}$.

Proof. Let $y_1, y_2, \cdots$ be any truly random sequence of independent samples from the uniform distribution on $0 \leqq y < 1$. We shall form the sequence $x_1, x_2, \cdots$ from the separate pairs y_1, y_2; y_3, y_4; $\cdots$ of the y-sequence. Let G be a fixed region in the unit square $0 \leqq u < 1, 0 \leqq v < 1$; let G^* be the complementary set. We define

$$\begin{aligned} x_{2n-1} &= y_{2n-1},\ x_{2n} = y_{2n} \quad \text{if} \quad (y_{2n-1}, y_{2n}) \in G \\ x_{2n-1} &= y_{2n},\ x_{2n} = y_{2n-1} \quad \text{if} \quad (y_{2n-1}, y_{2n}) \in G^*. \end{aligned} \tag{6.2}$$

For any G this transformation leaves the sequence x_n equidistributed, since for every $N = 1, 2, \cdots$

$$\Big| \sum_{\substack{a \leqq x_n < b \\ n \leqq N}} 1 - \sum_{\substack{a \leqq y_n < b \\ n \leqq N}} 1 \Big| \leqq 1. \tag{6.3}$$

For any G we shall compute the autocorrelation function $R(\tau)$ of the x-sequence and compute $\Pr(x_n > x_{n+1})$. We shall then choose G to make $R(\tau) = 0$ for $\tau \neq 0$ but make $\Pr(x_n > x_{n+1}) > \frac{1}{2}$.

Let g be the area of G; we assume $0 < g < 1$. Let α be the area of the intersection of G with the triangle $0 \leqq v < u < 1$, and let β be the area of the intersection of G with the triangle $0 \leqq u < v < 1$; thus $\alpha + \beta = g$. Let γ and δ be the moments

$$\gamma = \frac{1}{g} \iint_G \left(u - \frac{1}{2}\right) du\, dv, \qquad \delta = \frac{1}{g} \iint_G \left(v - \frac{1}{2}\right) du\, dv. \tag{6.4}$$

Let numbers g^*, α^*, β^*, γ^*, δ^* be defined analogously with respect to G^*. Thus $\alpha^* + \beta^* = g^*$, $\alpha + \alpha^* = 1/2$, $\beta + \beta^* = 1/2$, $g + g^* = 1$, and

$$g\begin{pmatrix}\gamma\\ \delta\end{pmatrix} + g^*\begin{pmatrix}\gamma^*\\ \delta^*\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

Let $P_n = (y_{2n-1}, y_{2n})(n = 1, 2, \cdots)$. Finally, let $a_n = x_n - \frac{1}{2}$, $b_n = y_n - \frac{1}{2}$.

First we compute $R(1) = E\, a_n a_{n+1}$. If n is odd, a_n and a_{n+1} come from the same point P_n. Therefore,

$$E\, a_{2n-1}a_{2n} = E\, b_{2n-1}b_{2n} = 0 \tag{6.5}$$

since the numbers $b_n = y_n - 1/2$ are uncorrelated with mean zero. If n is even, the numbers a_n, a_{n+1} come from two consecutive points P. Thus

$$\begin{aligned}(a_{2n}, a_{2n+1}) &= (b_{2n}, b_{2n+1}) && \text{if} \quad P_n \in G, P_{n+1} \in G\\ &= (b_{2n}, b_{2n+2}) && \text{if} \quad P_n \in G, P_{n+1} \in G^*\\ &= (b_{2n-1}, b_{2n+1}) && \text{if} \quad P_n \in G^*, P_{n+1} \in G\\ &= (b_{2n-1}, b_{2n+2}) && \text{if} \quad P_n \in G^*, P_{n+1} \in G^*.\end{aligned} \tag{6.6}$$

Therefore, as we shall explain directly,

$$Ea_{2n}a_{2n+1} = g^2\delta\gamma + gg^*\,\delta\delta^* + g^*g\gamma^*\gamma + g^{*2}\gamma^*\delta^*. \tag{6.7}$$

The first term $g^2\delta\gamma$ is simply the probability g^2 that both points P_n, P_{n+1} lie in G, multiplied by the mean values δ, γ of $b_{2n} = v - \frac{1}{2}$ in G and of $b_{2n+1} = u - \frac{1}{2}$ in G; the other three terms arise similarly. Using the identities $g^*\gamma^* = -g\gamma$, $g^*\delta^* = -g\delta$, we find from (6.7)

$$Ea_{2n}a_{2n+1} = -g^2(\gamma - \delta)^2. \tag{6.8}$$

Since

$$R(1) = Ea_na_{n+1} = \tfrac{1}{2}(Ea_{2n-1}a_{2n} + Ea_{2n}a_{2n+1})$$

we have from (6.5) and (6.8)

$$R(1) = -\tfrac{1}{2}g^2(\gamma - \delta)^2. \tag{6.9}$$

Next we compute $R(\tau)$ for $\tau = 2s = 2, 4, 6, \cdots$. We have

$$R(2s) = E(a_na_{n+2s}) = \tfrac{1}{2}E(a_{2n-1}a_{2n-1+2s} + a_{2n}a_{2n+2s}). \tag{6.10}$$

But

$$\begin{aligned}a_{2n-1}a_{2n-1+2s} &+ a_{2n}a_{2n+2s}\\ &= b_{2n-1}b_{2n-1+2s} + b_{2n}b_{2n+2s} && \text{if} \quad P_n \in G, P_{n+s} \in G\\ &= b_{2n-1}b_{2n+2s} + b_{2n}b_{2n-1+2s} && \text{if} \quad P_n \in G, P_{n+s} \in G^*\\ &= b_{2n}b_{2n-1+2s} + b_{2n-1}b_{2n+2s} && \text{if} \quad P_n \in G^*, P_{n+s} \in G\\ &= b_{2n}b_{2n+2s} + b_{2n-1}b_{2n-1+2s} && \text{if} \quad P_n \in G^*, P_{n+1} \in G^*.\end{aligned} \tag{6.11}$$

Therefore, $E(a_{2n-1}a_{2n-1+2s} + a_{2n}a_{2n+2s})$ equals

$$g^2(\gamma^2 + \delta^2) + gg^*(\gamma\delta^* + \delta\gamma^*) + g^*g(\delta^*\gamma + \gamma\delta^*) + g^{*2}(\delta^{*2} + \gamma^{*2}) = 2g^2(\gamma - \delta)^2.$$

Therefore,

$$R(2s) = g^2(\gamma - \delta)^2 \qquad (s = 1, 2, \cdots). \tag{6.12}$$

The cases $\tau = 2s + 1 \quad (s = 1, 2, \cdots)$ are slightly more complicated. We have

$$\begin{aligned} a_{2n-1}a_{2n+2s} &= b_{2n-1}b_{2n+2s} && \text{if} \quad P_n \in G, P_{n+s} \in G \\ &= b_{2n-1}b_{2n+2s-1} && \text{if} \quad P_n \in G, P_{n+s} \in G^* \\ &= b_{2n}b_{2n+2s} && \text{if} \quad P_n \in G^*, P_{n+s} \in G \\ &= b_{2n}b_{2n+2s-1} && \text{if} \quad P_n \in G^*, P_{n+s} \in G^*. \end{aligned}$$

But a term $a_{2n}a_{2n+2s+1}$ will involve a third point, P_{n+s+1} :

$$\begin{aligned} a_{2n}a_{2n+2s+1} &= b_{2n}b_{2n+2s+1} && \text{if} \quad P_n \in G, P_{n+s+1} \in G \\ &= b_{2n}b_{2n+2s+2} && \text{if} \quad P_n \in G, P_{n+s+1} \in G^* \\ &= b_{2n-1}b_{2n+2s+1} && \text{if} \quad P_n \in G^*, P_{n+s+1} \in G \\ &= b_{2n-1}b_{2n+2s+2} && \text{if} \quad P_n \in G^*, P_{n+s+1} \in G^*. \end{aligned}$$

Therefore, for $s = 1, 2, \cdots$,

$$\begin{aligned} 2R(2s + 1) &= Ea_{2n-1}a_{2n+2s} + Ea_{2n}a_{2n+2s+1} = g^2\gamma\delta + gg^*\gamma\gamma^* + g^*g\delta^*\delta + g^{*2}\delta^*\gamma^* \\ &\quad + g^2\delta\gamma + gg^*\delta\delta^* + g^*g\gamma^*\gamma + g^{*2}\gamma^*\delta^* = 2g^2(\gamma\delta - \gamma^2 - \delta^2 + \delta\gamma) \end{aligned} \tag{6.13}$$

$$R(2s + 1) = -g^2(\gamma - \delta)^2 \qquad (s = 1, 2, \cdots).$$

Having computed $R(\tau)$ for all $\tau \neq 0$, we shall compute $\Pr(x_n > x_{n+1})$. If n is odd, the numbers y_n, y_{n+1} are the coordinates u, v of the same point P_j, where $n = 2j - 1$. But

$$\begin{aligned} (x_{2j-1}, x_{2j}) &= (y_{2j-1}, y_{2j}) && \text{if} \quad P_j \in G \\ &= (y_{2j}, y_{2j-1}) && \text{if} \quad P_j \in G^*. \end{aligned}$$

Therefore, $x_{2j-1} > x_{2j}$ when $(u, v) = P_j \in G$ and $u > v$, or $P_j \in G^*$ and $v > u$. By the definitions of the areas α, β, α^*, β^*

$$\Pr(x_{2j-1} > x_{2j}) = \alpha + \beta^* = \alpha + \tfrac{1}{2} - \beta. \tag{6.14}$$

When $n = 2j$ is even, the numbers x_{2j}, x_{2j+1} come from different points P_j, P_{j+1}. We have

$$\Pr(x_{2j} > x_{2j+1}) = g^2A + gg^*B + g^*gC + g^{*2}D \tag{6.15}$$

where, if $P_j = (u_j, v_j) = (y_{2j-1}, y_{2j})$,

$$\begin{aligned} A &= \Pr(v_j > u_{j+1} \mid P_j \in G, \; P_{j+1} \in G) \\ B &= \Pr(v_j > v_{j+1} \mid P_j \in G, \; P_{j+1} \in G^*) \\ C &= \Pr(u_j > u_{j+1} \mid P_j \in G^*, P_{j+1} \in G) \\ D &= \Pr(u_j > v_{j+1} \mid P_j \in G^*, P_{j+1} \in G^*). \end{aligned} \tag{6.16}$$

We must now define four probability densities $\phi(u)$, $\psi(v)$, $\phi^*(u)$, $\psi^*(v)$. We define

$$
\begin{aligned}
\phi(u)\,du &= \Pr(u \leqq u' \leqq u + du \mid (u', v') \in G) \\
\psi(v)\,dv &= \Pr(v \leqq v' \leqq v + dv \mid (u', v') \in G)
\end{aligned}
\tag{6.17}
$$

or, equivalently,

$$
\begin{aligned}
\Phi(u) &= \int_0^u \phi(u')\,du' = \frac{1}{g} \iint_{\substack{(u',v') \in G \\ u' \leqq u}} du'\,dv' \\
\Psi(v) &= \int_0^v \psi(v')\,dv' = \frac{1}{g} \iint_{\substack{(u',v') \in G \\ v' \leqq v}} du'\,dv'.
\end{aligned}
\tag{6.18}
$$

The probability densities $\phi^*(u)$, $\psi^*(v)$ and their integrals Φ^*, Ψ^* are defined analogously for the region G^*. Thus

$$
g\phi(u) + g^*\phi^*(u) = 1, \qquad g\psi(v) + g^*\psi^*(v) = 1. \tag{6.19}
$$

From (6.16) we now compute

$$
\begin{aligned}
A &= \int_0^1 \int_0^{v_j} \psi(v_j)\phi(u_{j+1})\,du_{j+1}\,dv_j \\
A &= \int_0^1 \psi(v)\Phi(v)\,dv.
\end{aligned}
\tag{6.20}
$$

Similarly we compute

$$
\begin{aligned}
B &= \int_0^1 \psi(v)\Psi^*(v)\,dv \\
C &= \int_0^1 \phi^*(u)\Phi(u)\,du \\
D &= \int_0^1 \phi^*(u)\Psi^*(u)\,du.
\end{aligned}
\tag{6.21}
$$

From (6.15) we now find $\Pr(x_{2j} > x_{2j+1})$ equal to

$$
\int_0^1 [g^2\psi(t)\Phi(t) + gg^*\psi\Psi^* + g^*g\phi^*\Phi + g^{*2}\phi^*\Psi^*]\,dt. \tag{6.22}
$$

From (6.19) we see that the integrand equals

$$
\begin{aligned}
&g^2\psi\Phi + g\psi(t - g\Psi) + g\Phi(1 - g\phi) + (1 - g\phi)(t - g\Psi) \\
&\qquad = g^2 \frac{d}{dt}\left(\Psi\Phi - \frac{1}{2}(\Psi^2 + \Phi^2)\right) + g\frac{d}{dt}(t\Phi - t\Psi) + t + 2gt(\psi - \phi).
\end{aligned}
$$

Integration gives

$$\Pr(x_{2j} > x_{2j+1}) = \int_0^1 (t + 2gt(\psi - \phi))\,dt$$

$$(6.23)\qquad = \frac{1}{2} + 2gE(v - u \mid (u, v) \in G)$$

$$\Pr(x_{2j} > x_{2j+1}) = \frac{1}{2} + 2g(\delta - \gamma).$$

From (6.14) and (6.23) we conclude

$$(6.24)\qquad \Pr(x_n > x_{n+1}) = \tfrac{1}{2}(1 + \alpha - \beta) + g(\delta - \gamma).$$

Now we are ready to define G. We have shown for $s = 1, 2, \cdots$

$$(6.25)\qquad R(1) = -\tfrac{1}{2}g^2(\gamma - \delta)^2, \qquad R(2s) = g^2(\gamma - \delta)^2, \qquad R(2s + 1) = -g^2(\gamma - \delta)^2.$$

Therefore, it will suffice to pick G to be any region which has its centroid $(\gamma + \frac{1}{2}, \delta + \frac{1}{2})$ on the line $u = v$, but which has more area α to the right of $u = v$ than it has area β to the left of $u = v$. This we may do, for example, by letting G consist of two small circles K and k, where K has area $2\epsilon > 0$ and lies to the right of $u = v$, and where k has area ϵ and lies to the left of $u = v$. Let K have center O and k have center o. We require that the line segment connecting O to o pass through the center of the square $(\frac{1}{2}, \frac{1}{2})$, and that the distance from the center of the square to o be twice the distance to O. Then $\gamma = \delta$, $\alpha = 2\epsilon$, $\beta = \epsilon$,

$$(6.26)\qquad \Pr(x_n > x_{n+1}) = \tfrac{1}{2}(1 + \epsilon), \qquad R(\tau) = 0 \qquad (\tau = 1, 2, \cdots).$$

For example, if $0 < \epsilon < \pi/128$, we may let K and k be the circles

$$(6.27)\qquad \left(u - \frac{5}{8}\right)^2 + \left(v - \frac{3}{8}\right)^2 < \frac{2\epsilon}{\pi}, \qquad \left(u - \frac{1}{4}\right)^2 + \left(v - \frac{3}{4}\right)^2 < \frac{\epsilon}{\pi}.$$

This completes the proof of the theorem.

7. Sequential Equidistribution in Higher Dimensions. We have so far considered various sequential properties of sequences $x_1, x_2, \cdots$ equidistributed in the one-dimensional line segment C_1. But for many applications we must simulate random sequences in higher-dimensional cubes C_r; typically $r = 2$ or 3. For these applications we require sequences of r-dimensional vectors

$$(7.1)\qquad y^{(n)} = (y_1^{(n)}, y_2^{(n)}, \cdots, y_r^{(n)}) \qquad (n = 1, 2, \cdots)$$

which are equidistributed in C_r. One can discuss various extensions of the notion of equipartition. It might also be useful to discuss the autocorrelation function

$$(7.2)\qquad R(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{\rho=1}^{r} \left(y_\rho^{(n)} - \frac{1}{2}\right)\left(y_\rho^{(n+\tau)} - \frac{1}{2}\right) \qquad (\tau = 0, 1, \cdots)$$

to define the related spectral density $\phi(\omega)$, to extend Jagerman's results to higher dimensions, and to discuss higher-dimensional white sequences.

However, in this paper we shall consider extensions only of the notions of equidistribution by k's and of complete equidistribution. For any $k = 1, 2, \cdots$ we shall say that the sequence of r-dimensional vectors $y^{(n)}$ is equidistributed by k's if the sequence of $k \cdot r$ dimensional vectors

$$
\begin{aligned}
w^{(n)} &= (y^{(n)}, \cdots, y^{(n+k-1)}) \\
&= (y_1^{(n)}, \cdots, y_r^{(n)}, y_1^{(n+1)}, \cdots, y_r^{(n+1)}, \cdots, y_1^{(n+k-1)}, \cdots, y_r^{(n+k-1)})
\end{aligned}
\tag{7.3}
$$

is equidistributed in the $k \cdot r$ dimensional unit cube C_{kr}. The sequence $y^{(n)}$ is completely equidistributed if it is equidistributed by k's for all k.

If $x_1, x_2, \cdots$ is a truly random sequence of independent samples from the uniform distribution on C_1, then

$$
y^{(n)} = (x_{nr}, x_{nr+1}, \cdots, x_{nr+r-1}) \qquad (n = 1, 2, \cdots)
\tag{7.4}
$$

provides a truly random sequence of independent samples from the uniform distribution on C_r.

For any one-dimensional sequence x_n, random or deterministic, we define the r-dimensional "derived sequence" $z^{(n)}$ by (7.4). We shall investigate the derived sequences of certain one-dimensional equidistributed sequences.

We first ask whether a sequence x_n equidistributed by r's for some $r > 1$ has an equidistributed r-dimensional derived sequence. This is not true in general.

THEOREM 19. *There is a sequence $x_1, x_2, x_3, \cdots$ equidistributed by 2's in C_1 for which the 2-dimensional derived sequence*

$$
y^{(1)} = (x_2, x_3), \quad y^{(2)} = (x_4, x_5), \quad y^{(3)} = (x_6, x_7), \cdots
\tag{7.5}
$$

is not equidistributed in C_2.

Proof. We shall construct a sequence x_n for which the pairs (x_n, x_{n+1}) are equidistributed in C_2 but for which the alternate pairs (x_{2j}, x_{2j+1}) are not equidistributed. Let g_n be any sequence equidistributed by twos, e.g., $g_n = \{n^2\alpha\}$, α irrational. If we let I and II represent, respectively, the left and the right halves of the interval $0 \leqq x < 1$, we create from the g-sequence a sequence x_n which may be represented schematically as follows:

$$
x_n = \mathrm{I}, \mathrm{I}, \mathrm{II}, \mathrm{II}, \mathrm{I}, \mathrm{I}, \mathrm{II}, \mathrm{II}, \cdots .
\tag{7.6}
$$

To be precise, we define

$$
\begin{aligned}
&x_1 = \tfrac{1}{2}g_1, \quad x_2 = \tfrac{1}{2}g_2, \quad x_3 = \tfrac{1}{2} + \tfrac{1}{2}g_3, \quad x_4 = \tfrac{1}{2} + \tfrac{1}{2}g_4, \\
&x_5 = \tfrac{1}{2}g_5, \quad x_6 = \tfrac{1}{2}g_6, \quad x_7 = \tfrac{1}{2} + \tfrac{1}{2}g_7, \quad x_8 = \tfrac{1}{2} + \tfrac{1}{2}g_8, \text{ etc.}
\end{aligned}
\tag{7.7}
$$

The successive pairs (x_n, x_{n+1}) have the schematic representation

$$
(x_n, x_{n+1}) = (\mathrm{I}, \mathrm{I}), (\mathrm{I}, \mathrm{II}), (\mathrm{II}, \mathrm{II}), (\mathrm{II}, \mathrm{I}), \cdots .
\tag{7.8}
$$

Thus the pairs (x_n, x_{n+1}) are equidistributed in the four sub-squares of C_2:

$$
\begin{aligned}
(\mathrm{I}, \mathrm{II}) &= (0 \leqq u < \tfrac{1}{2}, \tfrac{1}{2} \leqq v < 1) \qquad (\mathrm{II}, \mathrm{II}) = (\tfrac{1}{2} \leqq u < 1, \tfrac{1}{2} \leqq v < 1) \\
(\mathrm{I}, \mathrm{I}) &= (0 \leqq u < \tfrac{1}{2}, 0 \leqq v < \tfrac{1}{2}) \qquad (\mathrm{II}, \mathrm{I}) \leqq (\tfrac{1}{2} \leqq u < 1, 0 \leqq v < \tfrac{1}{2}).
\end{aligned}
$$

Therefore, (x_n, x_{n+1}) is equidistributed in C_2. But the successive pairs (x_{2j}, x_{2j+1}) have the schematic representation

$$
(x_{2j}, x_{2j+1}) = (\mathrm{I}, \mathrm{II}), (\mathrm{II}, \mathrm{I}), (\mathrm{I}, \mathrm{II}), (\mathrm{II}, \mathrm{I}), \cdots .
\tag{7.9}
$$

Since the sub-squares (I, I) and (II, II) remain empty, the sequence (x_{2j}, x_{2j+1}) cannot be equidistributed in C_2. This completes the proof.

THEOREM 20. *For some* x_0 *let* $x_{n+1} = \{Nx_n + \theta\}$ $(n = 0, 1, \cdots)$, $N =$ *integer* $\geqq 2$. *For any such sequence the r-dimensional derived sequence* (7.4) *cannot be equidistributed in* C_r *for any* $r > 1$.

Proof. Suppose that for some $r > 1$ the derived sequence $y^{(n)}$ defined by (7.4) were equidistributed. From this assumption we shall deduce that the original sequence x_n is equidistributed by r's. Let h be the r-dimensional vector $h = (\theta, \theta, \cdots, \theta)$. Since N is an integer $\neq 0$, it is an immediate consequence of the Weyl criterion for equidistribution in C_r that the sequence

$$y^{(n,1)} = Ny^{(n)} + h \qquad (n = 1, 2, \cdots)$$

is also equidistributed modulo 1 in C_r. Consequently,

$$y^{(n,2)} = Ny^{(n,1)} + h \qquad (n = 1, 2, \cdots)$$

is equidistributed modulo 1 in C_r, et cetera; each of the r sequences $y^{(n)}, y^{(n,1)}, \cdots, y^{(n,r-1)}$ is equidistributed modulo 1 in C_r. But, since $Nx_k + \theta \equiv x_{k+1}$ modulo 1,

$$\begin{aligned} y^{(n)} &= (x_{nr}, x_{nr+1}, \cdots, x_{nr+r-1}) \\ y^{(n,1)} &\equiv (x_{nr+1}, x_{nr+2}, \cdots, x_{nr+r}) \qquad (\text{mod } 1) \\ &\cdots \\ y^{(n,r-1)} &\equiv (x_{nr+r-1}, x_{nr+r}, \cdots, x_{nr+2r-2}) \quad (\text{mod } 1). \end{aligned}$$

To show that the original sequence x_n is equidistributed by r's we must show that the sequence of vectors $z^{(k)} = (x_k, x_{k+1}, \cdots x_{k+r-1})(k = 1, 2, \cdots)$ is equidistributed in C_r. But

$$z^{(r)}, z^{(r+1)}, \cdots \equiv y^{(1)}, y^{(1,1)}, \cdots, y^{(1,r-1)}, y^{(2)}, y^{(2,1)}, \cdots, y^{(2,r-1)}, \cdots.$$

In general, for $k \geqq r$, if $y^{(n,0)} = y^{(n)}$,

$$z^{(k)} \equiv y^{(n,\rho)} \qquad (\rho = 0, \cdots, r-1; \quad n = 1, 2, \cdots; \quad k = nr + \rho).$$

Therefore, $z^{(k)}$ is equidistributed in C_r, and the Multiply sequence x_n is equidistributed by r's. But this is a contradiction to Theorem 4, which implies that no Multiply sequence is equidistributed by r's for any $r > 1$.

THEOREM 21. *Let* α *be irrational. Let*

$$(7.10) \quad x_n = \{\alpha n^p + c_1 n^{p-1} + c_2 n^{p-2} + \cdots + c_p\} \qquad (n = 1, 2, \cdots).$$

The r-dimensional derived sequence $y^{(n)}$ *is equidistributed by k's if and only if* $kr \leqq p$.

Proof. For $r = 1$ this theorem reduces to Theorem 11 in Section 4. The sequence of $y^{(n)}$ is equidistributed by k's when the sequence

$$(7.11) \qquad w^{(n)} = (x_{nr}, x_{nr+1}, \cdots, x_{nr+kr-1}) \qquad (n = 1, 2, \cdots)$$

is equidistributed in C_{kr}. We now proceed as in the proof of Theorem 11. Let $f(n) = \alpha n^p + \cdots + c_p$. Let $\Delta f(m) = f(m+1) - f(m)$. The Weyl criterion states that $w^{(n)}$ is equidistributed when

$$\lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} \exp 2\pi i \sum_{j=0}^{kr-1} h_j f(nr + j) = 0$$

for any integers $h_0, \cdots, h_{kr-1}$ not all zero, or, equivalently, when

$$\lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} \exp 2\pi i \sum_{j=0}^{kr-1} h_j' \Delta^j f(nr) = 0 \tag{7.12}$$

for any integers $h_0', \cdots, h_{kr-1}'$ not all zero.

If $kr > p$, we may choose $h_p' = 1, \quad h_j' = 0 \quad (j \neq p)$. Then for every $N = 1, 2, \cdots$

$$N^{-1}\sum_{n=1}^{N} \exp 2\pi i \sum_{j=0}^{kr-1} h_j'\Delta^j f(nr) = N^{-1}\sum_{n=1}^{N} \exp 2\pi i \Delta^p f(nr)$$

$$= N^{-1}\sum_{n=1}^{N} \exp 2\pi i\, p!\, r^p\alpha = \exp 2\pi i\, p!\, r^p\alpha \neq 0.$$

Therefore, $w^{(n)}$ is not equidistributed if $kr > p$. But if $kr \leqq p$, and if $h_j' = 0$ for $j < s$ but $h_s' \neq 0$, then $\sum h_j'\Delta^j f(nr)$ is a polynomial in n of degree $p - s \geqq 1$ with leading coefficient

$$\beta = p(p-1)\cdots(p-s+1)r^p\alpha h_s'.$$

Since β is irrational, we have the required zero limit (7.12). This completes the proof.

THEOREM 22. *For almost all $\theta > 1$, for any $r > 0$, the r-dimensional sequence $y^{(n)}$ derived from $x_n = \{\theta^n\}$ is completely equidistributed.*

Proof. We must show that for almost all $\theta > 1$ the sequence

$$w^{(n)} \equiv (\theta^{nr}, \theta^{nr+1}, \cdots, \theta^{nr+kr-1}) \pmod 1$$

is equidistributed in C_{kr}. The Weyl criterion requires

$$\lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} \exp 2\pi i \sum_{j=0}^{kr-1} h_j\theta^{nr+j} = 0 \tag{7.13}$$

for any integers $h_0, \cdots, h_{kr-1}$ not all zero, i.e.,

$$\lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} \exp 2\pi i p(\theta)\theta^{nr} = 0 \tag{7.14}$$

where $p(\theta) = \sum h_j\,\theta^j =$ a polynomial which is not identically zero. But by Theorem 14, if we set $M(n) = nr$ and $L = r$, the sequence $p(\theta)\theta^{nr} \quad (n = 1, 2, \cdots)$ is equidistributed modulo 1 for almost all $\theta > 1$. The relation (7.14), therefore, follows from the Weyl criterion applied to the one-dimensional sequence $p(\theta)\theta^{nr}$.

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