Matrix Assignments and an Associated Min Max Problem

By T. A. Porsching

1. Introduction. Consider an $n \times n$ matrix, $A = (a_{ij})$ of positive real numbers, and let Φ be the set of $n!$ permutations of the numbers $1, 2, \cdots, n$. An assignment T_ϕ is any set $\{a_{1\phi(1)}, a_{2\phi(2)}, \cdots, a_{n\phi(n)}\}$ of n elements of A with $\phi \in \Phi$. Furthermore, define the number μ by the relation,

$$\mu = \min_{\phi \epsilon \Phi} \max_{a_{ij} \epsilon T_\phi} a_{ij} .$$

We are concerned with an algorithm for determining μ which is more efficient than the obvious one of generating the $n!$ possible assignments then straightforwardly selecting μ. A method for demonstrating an assignment containing μ is also of concern, but such a method is easily evolved using the tools necessary to determine μ. Before proceeding we define a *nonzero column* of a set of r rows of A as a column which contains at least one nonzero element.

2. Determination of μ. Note that if R is the set consisting of the minimum elements of the rows and columns of A, $\mu \geqq \mu_0 = \max_{a_{ij} \epsilon R} a_{ij}$. This is clear if we remember that μ is the maximum of some assignment which contains an element from every row and column of A. In particular, if a_{ij} is the element of this assignment taken from the ith row and jth column of A, $\mu \geqq a_{ij} \geqq a_{ik}$ where $a_{ik} \in R$. The same is true of the jth column. Since this is true for $i, j = 1, 2, \cdots, n$, $\mu \geqq \mu_0$ as asserted. With this in mind we construct an $n \times n$ matrix A_0^* which has as its only nonzero elements the elements of R arranged as they were in A. If the a_{ij} are not all distinct, then all elements $\leqq \mu_0$ must also be inserted into A_0^*. Thus, the matrix A_0^* is simply the matrix A with all a_{ij} such that $a_{ij} > \mu_0$ replaced by zeros.

Now assume that it is possible to form an assignment from the nonzero elements of A_0^*. If the maximum element of this assignment is ν, then from the definition of μ, $\mu \leqq \nu$. But $\nu \leqq \mu_0$, so that $\mu \leqq \nu \leqq \mu_0 \leqq \mu$. This implies that $\mu = \nu = \mu_0$; that is, μ is the maximum element of A_0^*.

For the above conclusion it was necessary to assume that an assignment could be formed from the nonzero elements of A_0^*. Suppose, on the other hand, that every assignment of A_0^* contains at least one zero. Then clearly $\mu \geqq \mu_1 > \mu_0$, where μ_1 is the smallest element of A greater than μ_0. Now alter A_0^* by inserting in A_0^* the μ_1 of A arranged as they were in A. This gives a new matrix A_1^*. The same reasoning used on A_0^*, shows that if there is an assignment of A_1^* with no zero elements, then $\mu = \mu_1$, the largest element of A_1^*. In general, it is clear that if A_i^* is formed by the process of alteration described above, and if A_i^* is the first such altered matrix which has an assignment containing no zero elements, then μ

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equals the maximum element of A_i^*. Since μ always exists, such an A_i^* will eventually be found.

Clearly what is lacking in the above method is a relatively simple test on A_i^* to determine whether or not there exists an assignment of A_i^* containing no zeros. Fortunately, such a test exists and is essentially described in the following theorem.

THEOREM 1. *Let A be an $n \times n$ matrix of real numbers. Let S_r be a set of r rows of A. Let k equal the number of nonzero columns in S_r. Then there exists an assignment of A containing no zeros if and only if $r \leqq k$ for all S_r, $r = 1, 2, \cdots, n$.*

We shall prove this theorem by appealing to a more general theorem of Hall on complete systems of distinct representatives (CDR) [1]. Suppose

$$F_1, F_2, \cdots, F_m \tag{1}$$

is a finite system of subsets of a given set S. A CDR of (1) is a set of m *distinct* elements of S:

$$a_1, a_2, \cdots, a_m$$

such that $a_i \in F_i$. Hall has proven:

THEOREM 2. *In order that a CDR of* (1) *shall exist, it is necessary and sufficient that for each $k = 1, 2, \cdots, m$ any selection of k of the sets* (1) *shall contain between them at least k elements of S.*

We replace the nonzero elements of A by integers designating the column in which they lie and let S be the resulting set of distinct nonzero integers. With F_i as the set of nonzero integers belonging to the ith row of the new A, Theorem 1 follows immediately from Theorem 2.

In view of Theorem 1, the problem now becomes one of generating all of the sets S_r. This is solved by noting that Σ, the collection of all S_r, may be put in $1 - 1$ correspondence with the set Γ of $2^n - 1$ distinct, nonvoid combinations of the numbers $1, 2, \cdots, n$. The correspondence is the obvious one: $\{n_1, n_2, \cdots, n_r\} \in \Gamma \leftrightarrow \{\text{row } n_1, \text{row } n_2, \cdots, \text{row } n_r\} \in \Sigma$. The set Γ is extremely easy to generate on a binary computer since its members correspond in an obvious manner to the binary representation of the numbers $1, 2, \cdots, 2^n - 1$.

3. An Assignment for μ. Let A_i^* be the matrix which yielded μ. Then A_i^* possesses an assignment containg μ. Hence, there exists at least one μ such that when the row and column containing this μ are deleted from A_i^*, the reduced matrix so obtained, A_{i1}^*, has an assignment containing no zero elements. The elements of this assignment are the $n - 1$ remaining elements of the desired assignment. Any element of A_{i1}^* which does not appear in this assignment will not affect the result of Theorem 1 if set equal to zero. However, if A_{i1}^* is known to have an assignment containing no zeros and the zeroing of a particular element of A_{i1}^* implies that the conditions of Theorem 1 do not hold for this matrix, then the zeroed element must be an element of any assignment of A_{i1}^* which contains no zeros. This gives rise to the following procedure.

1. Sweep A_i^* setting its nonzero elements equal to zero one at a time, applying Theorem 1 after each zeroing. The first time the conditions of the theorem do not hold, remember the row and column of the last element set equal to zero and delete them from A_i^* to get A_{i1}^*.

2. Repeat Step 1 on $A_{i1}^*, A_{i2}^*, \cdots, A_{i(n-1)}^*$. If all remembering is done relative to A, then the rows and columns so remembered give the positions in A of n elements which constitute the desired assignment.

4. Conclusion. We conclude with a simple example illustrating the aspects of the preceding development.

For a 4×4 matrix the sets S_r are listed in the following table along with their binary analogs.

Decimal Number	Binary Equivalent	S_r Row Numbers	Decimal Number	Binary Equivalent	S_r Row Numbers
1	0001	1	9	1001	1, 4
2	0010	2	10	1010	2, 4
3	0011	1, 2	11	1011	1, 2, 4
4	0100	3	12	1100	3, 4
5	0101	1, 3	13	1101	1, 3, 4
6	0110	2, 3	14	1110	2, 3, 4
7	0111	1, 2, 3	15	1111	1, 2, 3, 4
8	1000	4			

If

$$A = \begin{pmatrix} 1 & 9 & 4 & 9 \\ 4 & 8 & 2 & 5 \\ 7 & 3 & 7 & 1 \\ 4 & 6 & 3 & 6 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix},$$

then

$$A_0^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{pmatrix}.$$

Since the theorem does not hold for $S_2 = \{2, 4\}$, $\mu > 3$. Note that if an S_r satisfies the condition of the theorem for A_i^*, it will also satisfy this condition for A_{i+j}^*, $j \geqq 0$. Thus, it is necessary only to consider an S_r until some A_i^* is found which meets the condition of the theorem. In the present example A_0^* is altered to give,

$$A_1^* = \begin{pmatrix} 1 & 0 & 4 & 0 \\ 4 & 0 & 2 & 0 \\ 0 & 3 & 0 & 1 \\ 4 & 0 & 3 & 0 \end{pmatrix},$$

and since $\{1, 2, 4\}$ fails, $\mu > 4$. The alteration of A_1^* yields,

$$A_2^* = \begin{pmatrix} 1 & 0 & 4 & 0 \\ 4 & 0 & 2 & 5 \\ 0 & 3 & 0 & 1 \\ 4 & 0 & 3 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & a_{13} & 0 \\ a_{21} & 0 & a_{23} & a_{24} \\ 0 & a_{32} & 0 & a_{34} \\ a_{41} & 0 & a_{43} & 0 \end{pmatrix},$$

for which no S_r fails and hence $\mu = 5$.

The process for obtaining a T_ϕ containing μ is now applied to A_2^*. Step 1 of this process immediately gives one element of T_ϕ as element a_{24} of A. Step 1 is completed by deleting row 2, column 4 from A_2^*. This leaves,

$$A_{21}^* = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 3 & 0 \\ 4 & 0 & 3 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{32} & 0 \\ a_{41} & 0 & a_{43} \end{pmatrix}.$$

Element a_{41} of this matrix is now set equal to zero as indicated in Step 1. This new form of A_{21}^* satisfies the theorem's hypothesis, so Step 1 is continued by setting a_{43} equal to zero. The matrix A_{21}^* now has the following appearance,

$$A_{21}^* = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The hypothesis of the theorem fails for this matrix and so another element of T_ϕ is a_{43}. Deleting row 3 and column 3 from this last matrix leaves,

$$A_{22}^* = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{32} \end{pmatrix}.$$

Step 1 is repeated on this matrix, and it is seen that T_ϕ contains a_{32} and that $A_{23}^* = (1) = (a_{11})$. From this it follows that the final element of T_ϕ is a_{11}. Therefore, one possible assignment is $T_\phi = \{a_{11}, a_{24}, a_{32}, a_{43}\}$.

As a final remark we note that with obvious simple modifications the algorithm developed here will also solve the analogous problem involving

$$\mu' = \max_{\phi \in \Phi} \min_{a_{ij} \in T_\phi} a_{ij}.$$

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1. P. Hall, "On representatives of subsets," *J. London Math. Soc.* 10, 1935, p. 26–30.

Formulas for Integrals of Products of Associated Legendre or Laguerre Functions

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1. Introduction. In this paper we derive, using a very simple technique, formulas for the integrals of products of Legendre functions,

$$(1) \qquad \int_{-1}^{1} P_{n_1}^{m_1}(x) P_{n_2}^{m_2}(x) \cdots P_{n_r}^{m_r}(x)\, dx,$$

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