The Calculation of Certain Dirichlet Series

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1. Introduction. We will be interested here in the computation of a class of Dirichlet series known as $L_a(s)$. They will be defined presently, and include such examples as

$$(1) \qquad L_3(n) = 1 - \frac{1}{5^n} + \frac{1}{7^n} - \frac{1}{11^n} + \frac{1}{13^n} - \frac{1}{17^n} + \frac{1}{19^n} - \frac{1}{23^n} + \cdots$$

and

$$(2) \qquad L_{-2}(n) = 1 - \frac{1}{3^n} - \frac{1}{5^n} + \frac{1}{7^n} + \frac{1}{9^n} - \frac{1}{11^n} - \frac{1}{13^n} + \frac{1}{15^n} + \cdots .$$

These, and some closely related series, arise in several number-theoretic investigations, including the distribution of primes into arithmetic progressions, the class number of binary quadratic forms, and the distribution of Legendre and Jacobi symbols. Our own immediate interest in them stems from their utility in the calculation of certain other number-theoretic constants. These latter include the h_a of references [1] and [2], the s_1 of reference [3], the constant 0.48762 of reference [4], and the constant $\frac{1}{2}C$ of reference [5]. The last of these illustrates our point, for when it was first presented by Bateman and Stemmler [6], it was given as 0.76; subsequently, in [5], the improved value 0.761 was presented; but with the aid of a short table of $L_3(s)$, and utilizing a formula analogous to that in [1, eq. (18), p. 323], it is fairly easy to compute

$$(3) \qquad \tfrac{1}{2}C = 0.7608578.$$

Similarly, while Ramanujan [11] gave a certain constant as $0.764 \cdots$, and G. Pall [12] gave another as $0.64 \cdots$, with the aid of short tables of $L_1(s)$ and $L_3(s)$, one may [13], by a trivial computation, obtain

$$b_1 = 0.764223654$$

and

$$b_3 = 0.638909405.$$

It may be argued that such precision as in (3) is not needed in these investigations. While that is irrelevant for our present paper, it is not inappropriate to mention briefly some counterarguments. Consider, for example, $P_2(N)$, the number of primes of the form $n^2 + 2$ for $1 \leqq n \leqq N$. The Hardy-Littlewood conjecture [1] claims that

$$(4) \qquad P_2(N) \sim 0.35653155 \operatorname{li}(N),$$

where $\operatorname{li}(N)$ is the logarithmic integral. While an examination of the evidence makes it almost certain that $P_2(N)$ is of the order of $\operatorname{li}(N)$, the heuristic argument in favor of (4) is not *that* convincing that one would be greatly astonished if the true coefficient were found to differ slightly from that conjectured. For empirical

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tests, therefore, fairly accurate coefficients are highly desirable. Secondly, one might (more ambitiously) attempt to investigate the second term, $S(N)$, in the presumed asymptotic series,

$$P_2(N) \sim 0.35653155 \operatorname{li}(N) + S(N),$$

but to do this an *accurate* subtraction of the leading term is absolutely essential. Such a second-term investigation arises from an assertion of Ramanujan concerning a certain $B(x)$. In effect, he states that

$$B(x) = \frac{b_1 x}{\sqrt{\log x}}\left[1 + \frac{1}{2\log x} + O\left(\frac{1}{\log^2 x}\right)\right].$$

But G. H. Hardy [11, p. 63], and Miss Stanley [14], disagree with the second term here, and go as far as to state (erroneously, [15]) that Ramanujan's approximation is no better than that of the leading term. Finally, consider a question such as this: Are there more primes (asymptotically speaking) of the form $n^2 + 2$ or of the form $n^2 + 6$? The Hardy-Littlewood conjecture [2] gives

$$\frac{P_2(N)}{P_6(N)} \sim 1.0000301,$$

but it is clear that this very slight excess of the first type could not have been found unless the pertinent constants were available to at least 5 or 6 decimals.

Let n be an odd integer, α an arbitrary integer, and (α/n) the Jacobi symbol. That is, if n is a product of odd primes, $n = \Pi p_i$, then $(\alpha/n) = \Pi(\alpha/p_i)$, where (α/p_i) is the Legendre symbol. If α is not prime to n, $(\alpha/n) = 0$.

Now, for every a, we define [1, p. 323]

$$L_a(s) = \sum_{\substack{\text{odd } n \\ n>0}} \left(\frac{-a}{n}\right)\frac{1}{n^s}. \tag{5}$$

The values of $(-a/n)$ are periodic in n and always either $+1$, -1, or 0. As examples we list the complete periods of these coefficients of n^{-s} (odd n) for twelve of our series.

(6)

$L_1(s)$: $+ -$

$L_2(s)$: $+ + - -$

$L_3(s)$: $+ 0 -$

$L_5(s)$: $+ + 0 + + - - 0 - -$

$L_6(s)$: $+ 0 + + 0 + - 0 - - 0 -$

$L_7(s)$: $+ - - 0 + + -$.

(7)

$L_{-1}(s)$: $+$

$L_{-2}(s)$: $+ - - +$

$L_{-3}(s)$: $+ 0 - - 0 +$

$L_{-5}(s)$: $+ - 0 - +$

$L_{-6}(s)$: $+ 0 + - 0 - - 0 - + 0 +$

$L_{-7}(s)$: $+ + - 0 + - - - - + 0 - + +$.

The reader may easily verify that $L_{\pm 4}(s) = L_{\pm 1}(s)$, and that $L_{\pm 8}(s) = L_{\pm 2}(s)$, but that

$$L_{\pm 9}(s) = \left(1 \pm \frac{1}{3^s}\right) L_{\pm 1}(s),$$

so that, for instance,

$$L_4(s): \ + -$$

$$L_8(s): \ + + - -$$

$$L_9(s): \ + \ 0 + - \ 0 \ -.$$

While $L_a(s)$ is an analytic function of the complex variable s that may be extended to the entire complex plane, our interest in it here will be confined to integral values of s.

Now $L_1(s)$ and $L_{-1}(s)$ have been tabulated and are well-known under the more common names: $L_1(s) = L(s)$ and $L_{-1}(s) = (1 - 2^{-s})\zeta(s)$. They need no further treatment here. Some of the other $L_a(s)$ have also been (partially) investigated long ago. Thus [7], [8] Glaisher's h_n is our $L_3(n)$, his q_n is our $L_{-2}(n)$, his p_n is our $L_2(n)$, and his t_n is our $L_{-3}(n)$. But this unsystematic notation of Glaisher is inadequate here, since we have defined $L_a(s)$ for infinitely many a. Further, it is clearly desirable for the notation to simply and unequivocally define the series in question. This is accomplished by the notation $L_a(s)$, as we have seen. (We may similarly criticize, at least for some purposes, the common notation $L(s, \chi)$. Unless χ, the particular *character* in question is fully defined, this notation is certainly ambiguous.)

Some of the $L_a(s)$ may be expressed in closed form. Thus, Glaisher gives

$$h_{2n+1} = \frac{1}{\sqrt{3}} \left(\frac{\pi}{3}\right)^{2n+1} \frac{H_n}{(2n)!},$$

where the coefficient H_n is generated by

$$\frac{3 \cos\left(\frac{1}{2} t\right)}{2 \cos\left(\frac{3}{2} t\right)} = \sum_{n=0}^{\infty} \frac{H_n t^{2n}}{(2n)!}.$$

Likewise, he gives

$$q_{2n} = \sqrt{2} \left(\frac{\pi}{4}\right)^{2n} \frac{Q_n}{(2n-1)!},$$

where

$$\frac{\sin t}{\cos(2t)} = \sum_{n=1}^{\infty} \frac{Q_n t^{2n-1}}{(2n-1)!}.$$

Here again, these individualized formulas and generating functions do not suffice for our more general needs.

We give below a brief presentation of the general theory of those $L_a(s)$ that may be expressed in closed form when s is an integer. This is a generalization and adaptation of the presentation given by Landau [9] for the evaluation of his $K(d)$.

As it develops, our treatment is somewhat simpler than Glaisher's, even for those examples that he did give. Thus [7, p. 64] Glaisher gives a two-term formula:

$$q_{2n} = \sqrt{2}(-1)^{n-1} \frac{2^{2n-1}\pi^{2n}}{(2n-1)!}\left\{A_{2n}\left(\frac{1}{8}\right) - \frac{1}{2^{2n}} A_{2n}\left(\frac{1}{4}\right)\right\},$$

while we obtain a one-term formula:

$$L_{-2}(s) = \frac{2}{\sqrt{2}} C_s\left(\frac{1}{8}\right).$$

Similarly, in [8, p. 102], we see a four-term formula:

$$\sqrt{6}\left(1 + \frac{1}{5^{2r}} - \frac{1}{7^{2r}} - \frac{1}{11^{2r}} - \frac{1}{13^{2r}} - \frac{1}{17^{2r}} + \frac{1}{19^{2r}} + \frac{1}{23^{2r}} + \cdots\right)$$
$$= (-1)^{r-1}\left\{b_{2r}\left(\frac{\pi}{12}\right) + b_{2r}\left(\frac{5\pi}{12}\right) - b_{2r}\left(\frac{7\pi}{12}\right) - b_{2r}\left(\frac{11\pi}{12}\right)\right\}$$

while we obtain the two-term fórmula:

$$L_{-6}(s) = \frac{2}{\sqrt{6}}\left\{C_s\left(\frac{1}{24}\right) + C_s\left(\frac{5}{24}\right)\right\}.$$

This relative simplicity stems from our use of $C_s(x)$, since, as we shall see, the series for this function, unlike those for the $A_{2n}(x)$ and $b_{2r}(x)$ above, involves only the powers of the successive *odd* integers.

In Sections 2 and 3 we give three tables of closed-form expressions for all the $L_a(s)$ that may be so expressed, with a range in s from 1 to 10, and for values of $a = \pm 2, \pm 3, \pm 5, \pm 6, \pm 7, \pm 10, \pm 11, \pm 13, \pm 14,$ and ± 15. In Section 4 we present the theory of $L_a(s)$ for integral values of $s < 1$.

In Section 5 we shall discuss computational techniques for evaluating those $L_a(s)$ *not* obtainable in closed form, as, for example, $L_3(2m)$. In that section we give twelve tables of $L_a(s)$ to 30D. Here $s = 1\ (1)\ 10$, and $a = \pm 1, \pm 2, \pm 3, \pm 6, \pm 9$, and ± 18. The first two tables are well-known, but are reproduced here for the reader's convenience. The last four are rational multiples of the first four, namely,

$$L_{\pm 9}(s) = \left(1 \pm \frac{1}{3^s}\right) L_{\pm 1}(s),$$

and

$$L_{\pm 18}(s) = \left(1 \mp \frac{1}{3^s}\right) L_{\pm 2}(s).$$

Taken together we have all four characters [9, p. 109] modulo 8:

$$L_{-1}(s):\quad ++++$$
$$L_{1}(s):\quad +-+-$$
$$L_{-2}(s):\quad +--+$$
$$L_{2}(s):\quad ++--;$$

all four characters modulo 12:

$$\begin{array}{ll} L_{-9}(s): & +0++0+ \\ L_{9}(s): & +0+-0- \\ L_{-3}(s): & +0--0+ \\ L_{3}(s): & +0-+0-; \end{array}$$

and all eight characters modulo 24:

$$\begin{array}{ll} L_{-9}(s): & +0++0++0++0+ \\ L_{9}(s): & +0+-0-+0+-0- \\ L_{-18}(s): & +0-+0--0+-0+ \\ L_{18}(s): & +0--0+-0++0- \\ L_{-6}(s): & +0+-0--0-+0+ \\ L_{6}(s): & +0++0+-0--0- \\ L_{-3}(s): & +0--0++0--0+ \\ L_{3}(s): & +0-+0-+0-+0-. \end{array}$$

It follows that the reader may readily compute the following arithmetical-progression Dirichlet series by simple linear combinations:

$$(8)\qquad \begin{aligned} \sum_{k=0}^{\infty} \frac{1}{(8k+1)^s} &= \tfrac{1}{4}[L_{-1}(s) + L_1(s) + L_{-2}(s) + L_2(s)], \\ \sum_{k=0}^{\infty} \frac{1}{(8k+3)^s} &= \tfrac{1}{4}[L_{-1}(s) - L_1(s) - L_{-2}(s) + L_2(s)], \\ \sum_{k=0}^{\infty} \frac{1}{(8k+5)^s} &= \tfrac{1}{4}[L_{-1}(s) + L_1(s) - L_{-2}(s) - L_2(s)], \\ \sum_{k=0}^{\infty} \frac{1}{(8k+7)^s} &= \tfrac{1}{4}[L_{-1}(s) - L_1(s) + L_{-2}(s) - L_2(s)]. \end{aligned}$$

Similarly, one may obtain

$$\sum_{k=0}^{\infty} \frac{1}{(12k+b)^s} \quad \text{or} \quad \sum_{k=0}^{\infty} \frac{1}{(24k+b)^s}$$

for b prime to 12 or 24, respectively, since, by weighting the character series by the coefficient that appears in the cth column in the foregoing arrays, one obtains, as the sum, 4, or 8, times the corresponding arithmetical-progression series for $b = 2c - 1$.

It was, of course, just such linear combinations, and their utility for his investigation of primes in arithmetical progressions, that led Dirichlet to invent these series.

As for

$$\sum_{k=0}^{\infty} \frac{1}{(mk+b)^s}$$

for $m = 8$, 12, or 24 and b not prime to m, those are also obtainable, with slightly more arithmetic. For example,

$$\sum_{k=0}^{\infty} \frac{1}{(8k+2)^s} = \frac{1}{2^{s+1}}[L_{-1}(s) + L_1(s)].$$

2. The Theory of the Closed Form Evaluations. The principles behind the closed-form evaluations are first, the trigonometric evaluation of the Jacobi symbol, and second, the subsequent evaluation of the derived Fourier series. Consider, for example:

$$\left(\frac{-5}{n}\right): \quad ++0++--0--, \text{ repeat.}$$

It is periodic, with a period of 20, and antisymmetric. It is therefore expressible as a (finite) Fourier sine series. In fact, if one evaluates

$$\sin\left[(2k+1)\frac{\pi}{10}\right] + \sin\left[(2k+1)\frac{3\pi}{10}\right]$$

for $k = 0, 1, 2, \cdots$, one obtains the repeating sequence:

$$+A,\ +A,\ 0,\ +A,\ +A,\ -A,\ -A,\ 0,\ -A,\ -A, \text{ repeat,}$$

where A is equal to $\frac{1}{2}\sqrt{5}$. The needed generalization for any (α/n) is obtained by the use of Gauss Sums.

Let d be an integer satisfying the two conditions [9, p. 219]:

$$\text{(9)} \qquad \begin{aligned} &\text{(a) } d \equiv 1 \pmod 4 \quad \text{or} \quad d \equiv 8 \text{ or } 12 \pmod{16}; \\ &\text{(b) } p^2 \nmid d \text{ for any odd prime } p. \end{aligned}$$

Some of the admissible values of d are $-24, -20, -8, -7, -4, -3, 5, 8, 12, 24, 28$. For any admissible d define the Kronecker symbol (d/n) for $n = 1, 2, 3, \cdots$, as follows [9, p. 70–72]. If n is odd, (d/n) is the Jacobi symbol; if n and d are both even, $(d/n) = 0$; and if n is even, d odd, then (d/n) equals the Jacobi symbol $(n/|d|)$.

For any admissible d we now have an identity [9, p. 221] that generalizes the Gauss Sum, namely

$$\text{(10)} \qquad \left(\frac{d}{n}\right) = \frac{1}{\sqrt{d}} \sum_{r=1}^{|d|} \left(\frac{d}{r}\right) e^{2\pi i n r/|d|}.$$

Inserting the appropriate Jacobi symbols on the right, the Kronecker symbol on the left is given as a linear combination of complex exponentials, that is, as a linear combination of sines or cosines. Using the abbreviations

$$\text{(11)} \qquad s_n(x) = \sin(2\pi n x), \qquad c_n(x) = \cos(2\pi n x),$$

the reader may verify the following evaluations of (d/n) for the 11 values of d mentioned above.

$$\text{(12)} \qquad \begin{aligned}
\left(\frac{-4}{n}\right) &= s_n\left(\frac{1}{4}\right) \\
\left(\frac{-8}{n}\right) &= \frac{1}{\sqrt{2}}\left[s_n\left(\frac{1}{8}\right) + s_n\left(\frac{3}{8}\right)\right] \\
\left(\frac{-3}{n}\right) &= \frac{2}{\sqrt{3}} s_n\left(\frac{1}{3}\right) \\
\left(\frac{-20}{n}\right) &= \frac{1}{\sqrt{5}}\left[s_n\left(\frac{1}{20}\right) + s_n\left(\frac{3}{20}\right) + s_n\left(\frac{7}{20}\right) + s_n\left(\frac{9}{20}\right)\right] \\
\left(\frac{-24}{n}\right) &= \frac{1}{\sqrt{6}}\left[s_n\left(\frac{1}{24}\right) + s_n\left(\frac{5}{24}\right) + s_n\left(\frac{7}{24}\right) + s_n\left(\frac{11}{24}\right)\right] \\
\left(\frac{-7}{n}\right) &= \frac{2}{\sqrt{7}}\left[s_n\left(\frac{1}{7}\right) + s_n\left(\frac{2}{7}\right) - s_n\left(\frac{3}{7}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
\left(\frac{8}{n}\right) &= \frac{1}{\sqrt{2}}\left[c_n\left(\frac{1}{8}\right) - c_n\left(\frac{3}{8}\right)\right] \\
\left(\frac{12}{n}\right) &= \frac{1}{\sqrt{3}}\left[c_n\left(\frac{1}{12}\right) - c_n\left(\frac{5}{12}\right)\right] \\
\left(\frac{5}{n}\right) &= \frac{2}{\sqrt{5}}\left[c_n\left(\frac{1}{5}\right) - c_n\left(\frac{2}{5}\right)\right] \\
\left(\frac{24}{n}\right) &= \frac{1}{\sqrt{6}}\left[c_n\left(\frac{1}{24}\right) + c_n\left(\frac{5}{24}\right) - c_n\left(\frac{7}{24}\right) - c_n\left(\frac{11}{24}\right)\right] \\
\left(\frac{28}{n}\right) &= \frac{1}{\sqrt{7}}\left[c_n\left(\frac{1}{28}\right) + c_n\left(\frac{3}{28}\right) - c_n\left(\frac{5}{28}\right)\right. \\
&\qquad \left. + c_n\left(\frac{9}{28}\right) - c_n\left(\frac{11}{28}\right) - c_n\left(\frac{13}{28}\right)\right].
\end{aligned} \tag{13}
$$

For our $L_a(s)$ we are only interested in the Jacobi symbol, that is, in such formulas for *odd* n. But for odd n we have two simplifications.

(a) On the right, since

$$c_n(\tfrac{1}{4}) = 0,$$

we have

$$
\begin{aligned}
s_n(\tfrac{1}{4} + y) &= s_n(\tfrac{1}{4} - y) \\
c_n(\tfrac{1}{4} + y) &= -c_n(\tfrac{1}{4} - y),
\end{aligned} \tag{14}
$$

and some terms may therefore be combined.

(b) On the left, if n is odd,

$$\left(\frac{4k}{n}\right) = \left(\frac{k}{n}\right). \tag{15}$$

Making these simplifications, we thus evaluate the following Jacobi symbols:

$$
\begin{aligned}
\left(\frac{-1}{n}\right) &= s_n\left(\frac{1}{4}\right) \\
\left(\frac{-2}{n}\right) &= \frac{2}{\sqrt{2}} s_n\left(\frac{1}{8}\right) \\
\left(\frac{-3}{n}\right) &= \frac{2}{\sqrt{3}} s_n\left(\frac{1}{3}\right) \\
\left(\frac{-5}{n}\right) &= \frac{2}{\sqrt{5}}\left[s_n\left(\frac{1}{20}\right) + s_n\left(\frac{3}{20}\right)\right] \\
\left(\frac{-6}{n}\right) &= \frac{2}{\sqrt{6}}\left[s_n\left(\frac{1}{24}\right) + s_n\left(\frac{5}{24}\right)\right] \\
\left(\frac{-7}{n}\right) &= \frac{2}{\sqrt{7}}\left[s_n\left(\frac{1}{7}\right) + s_n\left(\frac{2}{7}\right) - s_n\left(\frac{3}{7}\right)\right].
\end{aligned} \tag{16}
$$

$$\left(\frac{2}{n}\right) = \frac{2}{\sqrt{2}} c_n\left(\frac{1}{8}\right)$$

$$\left(\frac{3}{n}\right) = \frac{2}{\sqrt{3}} c_n\left(\frac{1}{12}\right)$$

$$\left(\frac{5}{n}\right) = \frac{2}{\sqrt{5}}\left[c_n\left(\frac{1}{5}\right) - c_n\left(\frac{2}{5}\right)\right] \tag{17}$$

$$\left(\frac{6}{n}\right) = \frac{2}{\sqrt{6}}\left[c_n\left(\frac{1}{24}\right) + c_n\left(\frac{5}{24}\right)\right]$$

$$\left(\frac{7}{n}\right) = \frac{2}{\sqrt{7}}\left[c_n\left(\frac{1}{28}\right) + c_n\left(\frac{3}{28}\right) - c_n\left(\frac{5}{28}\right)\right].$$

Substitution of these evaluations in (5) converts our Dirichlet series into Fourier series. (It is, of course, not merely a coincidence that the first rigorous work on the latter was done by Dirichlet.) We abbreviate

$$\begin{aligned} S_s(x) &= \sum_{k=0}^{\infty} \frac{\sin 2\pi(2k+1)x}{(2k+1)^s} \\ C_s(x) &= \sum_{k=0}^{\infty} \frac{\cos 2\pi(2k+1)x}{(2k+1)^s} \end{aligned} \tag{18}$$

and obtain

$$L_1(s) = S_s\left(\frac{1}{4}\right)$$

$$L_2(s) = \frac{2}{\sqrt{2}} S_s\left(\frac{1}{8}\right)$$

$$L_3(s) = \frac{2}{\sqrt{3}} S_s\left(\frac{1}{3}\right)$$

$$L_5(s) = \frac{2}{\sqrt{5}}\left[S_s\left(\frac{1}{20}\right) + S_s\left(\frac{3}{20}\right)\right]$$

$$L_6(s) = \frac{2}{\sqrt{6}}\left[S_s\left(\frac{1}{24}\right) + S_s\left(\frac{5}{24}\right)\right]$$

$$L_7(s) = \frac{2}{\sqrt{7}}\left[S_s\left(\frac{1}{7}\right) + S_s\left(\frac{2}{7}\right) - S_s\left(\frac{3}{7}\right)\right] \tag{19}$$

$$L_{-2}(s) = \frac{2}{\sqrt{2}} C_s\left(\frac{1}{8}\right)$$

$$L_{-3}(s) = \frac{2}{\sqrt{3}} C_s\left(\frac{1}{12}\right)$$

$$L_{-5}(s) = \frac{2}{\sqrt{5}}\left[C_s\left(\frac{1}{5}\right) - C_s\left(\frac{2}{5}\right)\right]$$

$$L_{-6}(s) = \frac{2}{\sqrt{6}}\left[C_s\left(\frac{1}{24}\right) + C_s\left(\frac{5}{24}\right)\right]$$

$$L_{-7}(s) = \frac{2}{\sqrt{7}}\left[C_s\left(\frac{1}{28}\right) + C_s\left(\frac{3}{28}\right) - C_s\left(\frac{5}{28}\right)\right].$$

The Fourier series $S_{2m-1}(x)$ and $C_{2m}(x)$ may, in turn, be evaluated in closed form for $m = 1, 2, 3, \cdots$.

Thus,

$$
(20)\qquad
\begin{aligned}
S_1(x) &= \frac{1}{4}\pi \\
C_2(x) &= \frac{1}{2}\pi^2\left(\frac{1}{4} - x\right) \\
S_3(x) &= \frac{1}{2}\pi^3\left(\frac{1}{2}x - x^2\right) \\
C_4(x) &= \pi^4\left(\frac{1}{96} - \frac{1}{4}x^2 + \frac{1}{3}x^3\right) \\
S_5(x) &= \frac{1}{6}\pi^5\left(\frac{1}{8}x - x^3 + x^4\right) \\
C_6(x) &= \frac{1}{3}\pi^6\left(\frac{1}{320} - \frac{1}{16}x^2 + \frac{1}{4}x^4 - \frac{1}{5}x^5\right) \\
S_7(x) &= \frac{1}{3}\pi^7\left(\frac{1}{160}x - \frac{1}{24}x^3 + \frac{1}{10}x^5 - \frac{1}{15}x^6\right) \\
C_8(x) &= \frac{2}{3}\pi^8\left(\frac{17}{105\cdot 1024} - \frac{1}{320}x^2 + \frac{1}{96}x^4 - \frac{1}{60}x^6 + \frac{1}{105}x^7\right) \\
S_9(x) &= \frac{1}{45}\pi^9\left(\frac{17}{7\cdot 256}x - \frac{1}{16}x^3 + \frac{1}{8}x^5 - \frac{1}{7}x^7 + \frac{1}{14}x^8\right) \\
C_{10}(x) &= \frac{1}{45}\pi^{10}\left(\frac{31}{63\cdot 1024} - \frac{17}{7\cdot 256}x^2 + \frac{1}{32}x^4 - \frac{1}{24}x^6 + \frac{1}{28}x^8 - \frac{1}{63}x^9\right).
\end{aligned}
$$

These formulas may be verified by the relations

$$
(21)\qquad
\begin{aligned}
S_s(x) &= -\frac{1}{2\pi}\frac{d}{dx}C_{s+1}(x), \\
C_s(x) &= \frac{1}{2\pi}\frac{d}{dx}S_{s+1}(x),
\end{aligned}
$$

and

$$C_{2n}(0) = -C_{2n}(\tfrac{1}{2}).$$

The latter relation fixes the constant term in $C_{2n}(x)$. We may note that $C_{2n}(0) = L_{-1}(2n)$. The formulas may clearly be extended to larger indices by integration.

One may therefore obtain, in closed form, $L_a(2n + 1)$ for positive a and $L_{-a}(2n)$ for positive a. Let us define $C_{a,n}$ and $D_{a,n}$ for positive a by the equations

$$
(22)\qquad
\begin{aligned}
L_a(2n + 1) &= \left(\frac{\pi}{a}\right)^{2n+1}\sqrt{a}\, C_{a,n} \\
L_{-a}(2n) &= \left(\frac{\pi}{a}\right)^{2n}\sqrt{a}\, D_{a,n}.
\end{aligned}
$$

We will give presently tables of such $C_{a,n}$ and $D_{a,n}$ up to $n = 4$ and 5, respectively.

To generalize the results of (19), we proceed as follows. Let $a = N^2b$, where b is not divisible by a square >1. Henceforth, the letter b here refers to such a number. Then it is readily seen that

$$L_a(s) = \prod_{p_i} \left[1 - \left(\frac{-b}{p_i}\right)\frac{1}{p_i^s}\right] L_b(s), \tag{23}$$

the product on the right being taken over each odd prime p_i that divides N. Now if

$$-b \equiv 1 \pmod 4$$

we evaluate the Kronecker symbol $(-b/n)$ by (10), while if

$$-b \equiv 2, 3 \pmod 4,$$

we evaluate $(-4b/n)$ by (10). The corresponding Jacobi symbol $(-b/n)$ is now inserted into $L_b(s)$ as before. We must distinguish four cases in our final result.

I. For $b > 0$, $b \equiv 3 \pmod 4$:

$$L_b(s) = \frac{2}{\sqrt{b}} \sum_{k=1}^{(b-1)/2} \left(\frac{k}{b}\right) S_s(k/b). \tag{24}$$

II. For $b < 0$, $-b \equiv 1 \pmod 4$:

$$L_b(s) = \frac{2}{\sqrt{-b}} \sum_{k=1}^{(-b-1)/2} \left(\frac{k}{-b}\right) C_s(k/-b). \tag{25}$$

III. For $b > 0$, $b \not\equiv 3 \pmod 4$:

$$L_b(s) = \frac{2}{\sqrt{b}} \sum_{\text{odd } k<b} \left(\frac{-b}{k}\right) S_s(k/4b). \tag{26}$$

IV. For $b < 0$, $-b \not\equiv 1 \pmod 4$:

$$L_b(s) = \frac{2}{\sqrt{-b}} \sum_{\text{odd } k<-b} \left(\frac{-b}{k}\right) C_s(k/-4b). \tag{27}$$

Using these and (20) and (22), we thus compute Tables 1 and 2.

3. Another Class of Closed Forms. There is another class of closed form evaluations, $L_{-b}(1)$ for $b > 0$. This is well known. We use the Fourier series

$$C_1(x) = \tfrac{1}{2} \log \cot \pi x, \tag{28}$$

and from (25) and (27) we thus obtain for

$$b > 0, \quad b \equiv 1 \pmod 4$$

$$L_{-b}(1) = \frac{1}{\sqrt{b}} \log \prod_{k=1}^{(b-1)/2} \{\cot(\pi k/b)\}^{\left(\frac{k}{b}\right)} \tag{29}$$

and for

$$b > 0, b \not\equiv 1 \pmod 4$$

$$L_{-b}(1) = \frac{1}{\sqrt{b}} \log \prod_{\text{odd } k<b} \{\cot(\pi k/4b)\}^{\left(\frac{b}{k}\right)} \tag{30}$$

TABLE 1. *Values of* $C_{a,n}$

a	n = 0	1	2	3	4
2	$\frac{1}{2}$	$\frac{3}{16}$	$\frac{19}{256}$	$\frac{307}{10240}$	$\frac{83579}{6881280}$
3	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{11}{24}$	$\frac{301}{720}$	$\frac{15371}{40320}$
5	1	$\frac{15}{8}$	$\frac{587}{128}$	$\frac{11851}{1024}$	$\frac{100822507}{3440640}$
6	1	$\frac{23}{8}$	$\frac{3985}{384}$	$\frac{1743623}{46080}$	$\frac{284922989}{2064384}$
7	$\frac{1}{2}$	4	$\frac{62}{3}$	$\frac{9271}{90}$	$\frac{644663}{1260}$
10	1	$\frac{79}{8}$	$\frac{39491}{384}$	$\frac{48224239}{46080}$	$\frac{109493813441}{10321920}$
11	$\frac{3}{2}$	$\frac{27}{2}$	$\frac{1275}{8}$	$\frac{155703}{80}$	$\frac{21370725}{896}$
13	1	$\frac{151}{8}$	$\frac{128657}{384}$	$\frac{265394311}{46080}$	$\frac{1018375291937}{10321920}$
14	2	$\frac{99}{4}$	$\frac{30195}{64}$	$\frac{23890611}{2560}$	$\frac{19125117629}{1032192}$
15	1	28	$\frac{1922}{3}$	$\frac{1314577}{90}$	$\frac{59937599}{180}$

For large values of b these formulas are not well adapted for easy and accurate computation. To supplement them, we now could follow the $K(d)$ theory in Landau [9], but the earlier pre-Kronecker theory of Gauss and Dirichlet is more direct for our purpose. (We register, in this connection, an objection to the frequent remark that Gauss was not wise in his choice of

$$f = Ax^2 + 2Bxy + Cy^2 \quad \text{and} \quad D = B^2 - AC \tag{31}$$

as the basis for his theory of binary quadratic forms, since for some purposes, such as our present one, and likewise for investigations in the distribution of Jacobi symbols, the Gauss theory is really simpler and more direct.) An exposition of this earlier theory in English was given by Mathews [10, Chapter VIII].

In [10, p. 238] we have directly

$$L_{-b}(1) = \frac{H(b) \log (T + U\sqrt{b})}{2\sqrt{b}} \tag{32}$$

TABLE 2. *Values of* $D_{a,n}$

a	n				
	1	2	3	4	5
2	$\frac{1}{4}$	$\frac{11}{96}$	$\frac{361}{7680}$	$\frac{24611}{1290240}$	$\frac{2873041}{371589120}$
3	$\frac{1}{2}$	$\frac{23}{48}$	$\frac{1681}{3840}$	$\frac{257543}{645120}$	$\frac{67637281}{185794560}$
5	1	$\frac{17}{6}$	$\frac{871}{120}$	$\frac{92777}{5040}$	$\frac{16922791}{362880}$
6	$\frac{3}{2}$	$\frac{87}{16}$	$\frac{25361}{1280}$	$\frac{15540167}{215040}$	$\frac{1813875289}{6881280}$
7	2	$\frac{159}{16}$	$\frac{44461}{960}$	$\frac{37040933}{161280}$	$\frac{52955730841}{46448640}$
10	$\frac{7}{2}$	$\frac{1577}{48}$	$\frac{1264807}{3840}$	$\frac{307191791}{92160}$	$\frac{6273958190407}{185794560}$
11	$\frac{7}{2}$	$\frac{2153}{48}$	$\frac{2130727}{3840}$	$\frac{627658799}{92160}$	$\frac{15515203176007}{185794560}$
13	5	$\frac{493}{6}$	$\frac{33463}{24}$	$\frac{120190933}{5040}$	$\frac{29632170055}{72576}$
14	5	$\frac{2503}{24}$	$\frac{802729}{384}$	$\frac{13406231743}{322560}$	$\frac{15337101636817}{18579456}$
15	6	$\frac{537}{4}$	$\frac{978941}{320}$	$\frac{3749253437}{53760}$	$\frac{2735126857009}{1720320}$

where $T + U\sqrt{b}$ is the smallest number >1 such that

$$T^2 - bU^2 = 1, \tag{33}$$

and $H(b)$ is the number of classes of properly primitive forms (31) for which $D = b$. Mathews also gives formulas similar to our (29) and (30), but slightly more complicated [10, p. 251–252].

The T and U of (33) are easily computed, and if one can obtain the integer $H(b)$, (32) would suffice. But if $H(b)$ is not available, one can proceed by the following combined operation. *Estimate* the logarithm on the right side of (29) or (30) with sufficient accuracy to identify the integer $H(b)$ by

$$H(b) = 2\,\frac{\text{logarithm in (29) or (30)}}{\log\,(T + U\sqrt{b})}. \tag{34}$$

Then compute $L_{-b}(1)$ *accurately* by (32). In Table 3, equation (32) is written as

$$L_{-b}(1) = \frac{\log\,(t + u\sqrt{b})}{\sqrt{b}}, \tag{35}$$

TABLE 3

b	$t + u\sqrt{b}$	b	$t + u\sqrt{b}$
2	$1 + \sqrt{2}$	10	$19 + 6\sqrt{10}$
3	$2 + \sqrt{3}$	11	$10 + 3\sqrt{11}$
5	$2 + \sqrt{5}$	13	$18 + 5\sqrt{13}$
6	$5 + 2\sqrt{6}$	14	$15 + 4\sqrt{14}$
7	$8 + 3\sqrt{7}$	15	$31 + 8\sqrt{15}$

where

$$t + u\sqrt{b} = (T + U\sqrt{b})^{H(b)/2}. \tag{36}$$

If $H(b)$ is odd, say for $b = 2, 5$, then

$$t^2 - bu^2 = -1;$$

otherwise

$$t^2 - bu^2 = 1.$$

One can, of course, compute $t + u\sqrt{b}$ by (29) or (30). An interesting and useful transformation of (29) is given when $b > 0$, $b \equiv 1 \pmod 4$ by

$$t + u\sqrt{b} = \frac{2^{\varphi(b)/2}}{F(b)} \prod_{k=1}^{(b-1)/2} \left\{1 + \left(\frac{k}{b}\right) \cos \frac{2\pi k}{b}\right\}. \tag{29a}$$

Here $\varphi(b)$ is Euler's function, and $F(b) = \sqrt{b}$ or 1, according as b is prime or not. Similarly, when $b > 0$, $b \not\equiv 1 \pmod 4$,

$$t + u\sqrt{b} = 2^{\varphi(4b)/4} \prod_{\text{odd } k<b} \left\{1 + \left(\frac{b}{k}\right) \cos \frac{\pi k}{2b}\right\}. \tag{30a}$$

For example, for $b = 15$,

$$31 + 8\sqrt{15} = 16\,(1 + \cos 6°)(1 + \cos 42°)(1 + \cos 66°)(1 - \cos 78°).$$

4. Nonpositive Arguments. To compute $L_a(s)$ for s an integer <1 we need an analytic continuation of the series in (5). In [16] Landau gives, in effect, a functional equation for certain Dirichlet series $L(s, \chi)$. If χ is a real, "eigenlichter" character modulo $k > 2$, we have

$$L(1 - s, \chi) = L(s, \chi) \left(\frac{k}{\pi}\right)^{s-1/2} \frac{\Gamma\left(\frac{s+\alpha}{2}\right)}{\Gamma\left(\frac{1-s+\alpha}{2}\right)} \tag{37}$$

where $\alpha = 0$ if $\chi(-1) = 1$ and $\alpha = 1$ is $\chi(-1) = -1$. In the first case we therefore have

$$L(1 - s, \chi) = L(s, \chi) \frac{2}{\sqrt{k}} \Gamma(s) \left(\frac{k}{2\pi}\right)^s \cos \frac{\pi s}{2}, \tag{38}$$

and in the second,

$$L(1 - s, \chi) = L(s, \chi) \frac{2}{\sqrt{k}} \Gamma(s) \left(\frac{k}{2\pi}\right)^s \sin \frac{\pi s}{2}. \tag{39}$$

For the $L_b(s)$ in our Case III, given by equation (26), equation (39) is valid with $k = 4b$; and in our Case IV, given by equation (27), equation (38) is valid with $k = -4b$. For our Cases I and II we must make our $L_b(s)$ depend upon appropriate related functions. In Case I, $b = 4m + 3$, since

$$L_b(s) = \sum_{\substack{\text{odd } n \\ n>0}} \left(\frac{-4m - 3}{n}\right) \frac{1}{n^s} = \sum_{\substack{\text{odd } n \\ n>0}} \left(\frac{n}{4m + 3}\right) \frac{1}{n^s}$$

we utilize

$$L_b(s) = \left[1 - \left(\frac{2}{4m + 3}\right) \frac{1}{2^s}\right] \sum_{n=1}^{\infty} \left(\frac{n}{4m + 3}\right) \frac{1}{n^s}. \tag{40}$$

For example,

$$L_3(s) = \left(1 + \frac{1}{2^s}\right) \sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{1}{n^s}$$

and

$$L_7(s) = \left(1 - \frac{1}{2^s}\right) \sum_{n=1}^{\infty} \left(\frac{n}{7}\right) \frac{1}{n^s}.$$

In (40), the sum on the right, call it $l_b(s)$, satisfies (39) with $k = b$. Similarly, in Case II, $b = -(4m + 1)$,

$$L_b(s) = \left[1 - \left(\frac{2}{4m + 1}\right) \frac{1}{2^s}\right] \sum_{n=1}^{\infty} \left(\frac{n}{4m + 1}\right) \frac{1}{n^s}, \tag{41}$$

and now $l_b(s)$ satisfies (38) with $k = -b$.

Thus, it follows that

$$L_b(0) = 0$$

in Cases II and IV, and also in Case I if $b = 8m + 7$. More generally,

$$L_b(-2n) = 0 \qquad (n \geqq 0)$$

in Cases II and IV. Likewise,

$$L_b(1 - 2n) = 0 \qquad (n \geqq 1)$$

in Cases I and III.

For the remaining nonpositive arguments, we utilize (22) and the foregoing, and obtain

Case I

$$L_b(-2n) = (-1)^n (2n)! \frac{2 - \left(\frac{2}{b}\right) 2^{2n+1}}{2^{2n+1} - \left(\frac{2}{b}\right)} C_{b,n} . \tag{42}$$

Case II

$$(43) \qquad L_b(1-2n) = (-1)^n(2n-1)!\,\frac{2-\left(\frac{2}{-b}\right)2^{2n}}{2^{2n}-\left(\frac{2}{-b}\right)}\,D_{-b,n}\,.$$

Case III

$$(44) \qquad L_b(-2n) = (-1)^n(2n)!\,2^{2n+1}C_{b,n}\,.$$

Case IV

$$(45) \qquad L_b(1-2n) = (-1)^n(2n-1)!\,2^{2n}D_{-b,n}\,.$$

Therefore, $L_b(s)$ is rational for all integers $s < 1$. For example, in Case I, $b = 3$, we have $L_3(0) = \frac{2}{3}$, $L_3(-1) = 0$, $L_3(-2) = -\frac{10}{9}$, $L_3(-3) = 0$, $L_3(-4) = \frac{34}{3}$, etc.

5. Numerical Evaluation of the Dirichlet Series. In Tables 4–15 we present 30D values of $L_{\pm a}(s)$ for $a = 1, 2, 3, 6, 9, 18$ and $s = 1(1)10$. The values of $L_{\pm 1}(s)$ have been included for ease of reference; they have been abridged from 50D values given by Liénard [17], who designated them as u_n and U_n, respectively.

From these data the entries in Tables 12 and 13 were derived by use of the relation

$$L_{\pm 9}(s) = (1 \pm 3^{-s})L_{\pm 1}(s),$$

which we have already noted.

Formulas (22) in conjunction with the appropriate coefficients presented in Tables 1 and 2 were used to evaluate $L_a(2n+1)$ and $L_{-a}(2n)$ to at least 35D when $a = 2, 3$, and 6. The requisite decimal approximations to powers of π were obtained from a manuscript table [18] of the second author.

Numerical values of $L_{-a}(1)$ were computed from equation (35) in conjunction with the corresponding entries in Table 3.

Evaluation of $L_a(2n)$ and $L_{-a}(2n+1)$ was accomplished numerically by means of the following relations:

$$(46) \qquad L_2(s) + L_{-2}(s) + L_1(s) + L_{-1}(s) = 4\sum_{k=0}^{\infty}(8k+1)^{-s},$$

$$(47) \qquad L_3(s) + L_{-3}(s) + L_9(s) + L_{-9}(s) = 4\sum_{k=0}^{\infty}(12k+1)^{-s},$$

$$(48) \qquad L_6(s) + L_{-6}(s) + L_{18}(s) + L_{-18}(s) = 4\sum_{k=0}^{\infty}(-1)^k(12k+1)^{-s},$$

where $L_{\pm 18}(s) = (1 \mp 3^{-s})L_{\pm 2}(s)$, as noted previously.

The right members of equations (46) and (47) were computed to 37D by means of the series

$$(49) \qquad \sum_{k=0}^{\infty}(nk+1)^{-s} = 1 + (n+1)^{-s} + (2n+1)^{-s} + \sum_{r=0}^{\infty}(-1)^r\binom{s+r-1}{s-1}S''_{s+r}\,n^{-s-r},$$

TABLE 4. *Values of* $L_1(s)$

s	
1	0. 78539 81633 97448 30961 56608 45820
2	0. 91596 55941 77219 01505 46035 14932
3	0. 96894 61462 59369 38048 36348 45847
4	0. 98894 45517 41105 33610 84226 33228
5	0. 99615 78280 77088 06400 63193 68631
6	0. 99868 52222 18438 13544 16007 87860
7	0. 99955 45078 90539 90949 63465 49899
8	0. 99984 99902 46829 65633 80670 59240
9	0. 99994 96841 87220 08982 13588 73294
10	0. 99998 31640 26196 87740 55407 29958

TABLE 5. *Values of* $L_{-1}(s)$

s	
1	∞
2	1. 23370 05501 36169 82735 43113 74985
3	1. 05179 97902 64644 99972 47708 91323
4	1. 01467 80316 04192 05454 62534 65507
5	1. 00452 37627 95139 61613 35103 15005
6	1. 00144 70766 40942 12190 64785 87138
7	1. 00047 15486 52376 55475 51116 31492
8	1. 00015 51790 25296 11930 29872 49296
9	1. 00005 13451 83843 77259 28179 00543
10	1. 00001 70413 63044 82548 81839 02300

TABLE 6. *Values of* $L_2(s)$

s	
1	1. 11072 07345 39591 56175 39702 47515
2	1. 06473 41710 43503 37039 28274 51462
3	1. 02772 25859 36858 56787 92566 18002
4	1. 01050 89405 73942 75298 78982 07302
5	1. 00375 56863 95655 01098 49997 54471
6	1. 00130 14424 54434 07196 30063 55517
7	1. 00044 34746 05655 00357 30275 05364
8	1. 00014 97085 46475 12110 59948 71805
9	1. 00005 02713 77482 72819 46630 55508
10	1. 00001 68294 64626 37816 45673 18164

TABLE 7. *Values of* $L_{-2}(s)$

s	
1	0. 62322 52401 40230 51339 40200 80251
2	0. 87235 80249 54859 94176 96951 17021
3	0. 95838 04545 63094 56205 16694 02862
4	0. 98654 28606 93970 50390 15344 90617
5	0. 99563 38563 12967 45634 22177 17128
6	0. 99857 39719 53530 54767 02705 16107
7	0. 99953 13156 79375 57755 04092 58244
8	0. 99984 52154 79225 60046 28798 89477
9	0. 99994 87096 11033 21295 58121 22854
10	0. 99998 29662 95349 76661 52219 07503

TABLE 8. *Values of* $L_3(s)$

s	
1	0. 90689 96821 17108 92529 70391 28821
2	0. 97662 80161 20607 87108 39842 87030
3	0. 99452 67882 18839 83883 59401 56480
4	0. 99877 72859 31944 00423 42612 85388
5	0. 99973 56076 48751 73899 50286 94789
6	0. 99994 41189 29516 44561 32463 64813
7	0. 99998 83774 09405 78786 21913 68362
8	0. 99999 76099 35612 82784 77901 86693
9	0. 99999 95124 45469 96620 68244 94259
10	0. 99999 99011 08484 72908 32293 15190

TABLE 9. *Values of* $L_{-3}(s)$

s	
1	0. 76034 59963 00946 34753 10942 54880
2	0. 94970 31262 94009 39526 34984 91746
3	0. 99004 00194 38159 94979 18167 76863
4	0. 99807 15998 37928 68732 97096 98120
5	0. 99962 84925 65847 67855 07322 81954
6	0. 99992 82178 25104 42180 14330 56231
7	0. 99998 60498 13929 33627 63518 45913
8	0. 99999 72722 38930 94714 67818 74964
9	0. 99999 94637 26660 51276 78505 04087
10	0. 99999 98941 05047 33421 17808 98652

TABLE 10. *Values of* $L_6(s)$

s	
1	1. 28254 98301 61864 09554 40363 59671
2	1. 05780 66132 11504 42946 64424 68367
3	1. 01090 26642 80502 41641 16825 04567
4	1. 00203 12462 48522 77528 41975 01406
5	1. 00038 19292 73761 64901 00617 72957
6	1. 00007 27944 52888 72917 79169 55577
7	1. 00001 40460 50564 39796 11536 33400
8	1. 00000 27369 27970 92548 78370 64745
9	1. 00000 05370 99012 55183 59109 36865
10	1. 00000 01059 70764 53154 51588 22095

TABLE 11. *Values of* $L_{-6}(s)$

s	
1	0. 93588 13101 03570 11048 69091 59266
2	1. 00731 22810 74837 31529 16284 36796
3	1. 00394 16231 50056 32071 76855 49326
4	1. 00108 13817 86318 35741 13239 97815
5	1. 00025 15406 94224 67683 00119 47776
6	1. 00005 47188 75849 30454 37286 31353
7	1. 00001 15176 18270 80347 48646 27001
8	1. 00000 23805 77479 83393 81482 96510
9	1. 00000 04866 96165 39056 96890 88342
10	1. 00000 00988 13760 96521 79921 82628

TABLE 12. *Values of* $L_9(s)$

s	
1	1. 04719 75511 96597 74615 42144 61093
2	1. 01773 95490 85798 90561 62261 27703
3	1. 00483 30405 65271 95013 11768 77175
4	1. 00115 37437 37909 10569 00080 97836
5	1. 00025 72430 07446 45110 09955 80025
6	1. 00005 51607 94869 46347 37566 18845
7	1. 00001 15515 61271 75216 18684 27608
8	1. 00000 23831 73250 45037 19548 91440
9	1. 00000 04868 94337 25794 05389 45380
10	1. 00000 00988 28886 61299 59386 29004

TABLE 13. *Values of* $L_{-9}(s)$

s	
1	∞
2	1. 09662 27112 32150 95764 82767 77764
3	1. 01284 42424 77065 55529 05201 17570
4	1. 00215 11423 25127 95510 74108 30131
5	1. 00038 99201 49892 12800 12736 47042
6	1. 00007 33495 12490 89814 52900 01970
7	1. 00001 40856 67167 42052 79716 62753
8	1. 00000 27395 83286 47197 49422 88581
9	1. 00000 05373 11812 89092 98298 99836
10	1. 00000 01059 86639 41566 20143 11216

TABLE 14. *Values of* $L_{18}(s)$

s	
1	0. 74048 04896 93061 04116 93134 98343
2	0. 94643 03742 60891 88479 36244 01299
3	0. 98965 87864 57715 65795 78026 69187
4	0. 99803 35215 54511 36097 57019 33138
5	0. 99962 50045 58635 85456 11931 71119
6	0. 99992 79150 98529 49847 60886 51326
7	0. 99998 60244 57229 92126 68669 98961
8	0. 99999 72699 38252 82646 78138 02628
9	0. 99999 94635 60006 86157 22886 88640
10	0. 99999 98940 91809 48666 12706 56623

TABLE 15. *Values of* $L_{-18}(s)$

s	
1	0. 83096 69868 53640 68452 53601 07001
2	0. 96928 66943 94288 82418 85501 30024
3	0. 99387 60269 54320 28657 21016 02968
4	0. 99872 24021 84019 52246 82201 01612
5	0. 99973 11149 80922 05492 79881 60408
6	0. 99994 37579 23288 47709 09430 40820
7	0. 99998 83487 45529 84164 62256 31933
8	0. 99999 76076 77896 41674 09568 41145
9	0. 99999 95122 68636 78117 26975 47440
10	0. 99999 99010 94690 91294 73632 68439

where $S_t'' = \zeta(t) - 1 - 2^{-t}$. Thus, the required values of S_t'' were obtained from tables of $\zeta(t)$, or S_t, given by Liénard [17].

The right member of equation (48) was expeditiously evaluated by means of the relation

$$\sum_{k=0}^{\infty} (-1)^k(12k+1)^{-s} = 2\sum_{k=0}^{\infty} (24k+1)^{-s} - \sum_{k=0}^{\infty} (12k+1)^{-s}.$$

A partial check on the accuracy of these tables was made by use of the relations

$$L_{-2}(s) + L_{-1}(s) - L_2(s) - L_1(s) = 4\sum_{k=1}^{\infty} (8k-1)^{-s} \tag{50}$$

$$L_{-3}(s) + L_{-9}(s) - L_3(s) - L_9(s) = 4\sum_{k=1}^{\infty} (12k-1)^{-s} \tag{51}$$

$$\begin{aligned} L_{-3}(s) + L_{-6}(s) + L_{-9}(s) + L_{-18}(s) - L_3(s) - L_6(s) - L_9(s) - L_{18}(s) \\ = 8\sum_{k=1}^{\infty} (24k-1)^{-s}. \end{aligned} \tag{52}$$

These check relations are especially easy to apply because the right members involve the same individual terms (except for sign) as the right members of equations (46) and (47) and the series $\sum (24k+1)^{-s}$, when evaluated by equation (49).

These check relations were satisfied to within a unit in the thirty-third decimal place when the corresponding data on the work sheets were successively substituted therein.

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