Tabulation of the Functions $\frac{\partial I_\nu(z)}{\partial \nu}$, $\nu = \pm\frac{1}{3}$

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1. Introduction. Partial derivatives of modified Bessel functions with respect to the "index" variable occur in a number of investigations. A useful illustration is provided by a variation on Watson's integral [1]

$$(1) \qquad \int_0^{\pi/2} d\theta\, K_0(2z\cos\theta)\cos(2\nu\theta) = -\frac{\pi}{4}\left\{I_{-\nu}(z)\frac{\partial}{\partial\nu}I_\nu(z) + I_\nu(z)\frac{\partial}{\partial\nu}I_{-\nu}(z)\right\};$$

where K_0 denotes a modified Bessel function of the second kind.

Oberhettinger [2] has previously discussed the degenerate cases $\partial I_\nu(z)/\partial\nu$, $\nu = \pm\frac{1}{2}$. The related problem of computing $\partial J_\nu(z)/\partial\nu$ has been considered by Lee and Radosevich [3], who also present brief numerical tables. References to earlier work may be found in both of these papers.

In the present paper we describe a tabulation of $\partial I_\nu(z)/\partial\nu$ to 4D, for $\nu = \pm\frac{1}{3}$, and $z = .01(.01)5.00$. This should be adequate for calculations requiring moderate precision. An analytical error discussion of sufficient generality to cover extensions to higher precision and other values of ν is also included. We append an abridgment of this table, corresponding to $z = .01(.01)1.00(.05)5.00$.

2. Analytical Preliminaries. We begin with the defining series representation

$$(2) \qquad I_\nu(z) = (z/2)^\nu \sum_{m=0}^{\infty} \frac{(z/2)^{2m}}{m!\,\Gamma(\nu+m+1)},$$

and immediately note the two limiting cases:

$$(3) \qquad I_\nu(z) \cong \frac{(z/2)^\nu}{\Gamma(\nu+1)}, \quad |z| \ll 1;$$

and

$$(4) \qquad I_\nu(z) \cong e^z/\sqrt{2\pi z}, \quad |z| \gg 1, \quad |z| \gg |\nu|.$$

By differentiation of (2) we obtain

$$(5) \qquad \frac{\partial I_\nu(z)}{\partial\nu} = \ln(z/2)\, I_\nu(z) - (z/2)^\nu \sum_{m=0}^{\infty} \frac{(z/2)^{2m}}{m!} \frac{\psi(\nu+m+1)}{\Gamma(\nu+m+1)};$$

which in turn yields the limiting case

$$(6) \qquad \frac{\partial I_\nu(z)}{\partial\nu} \cong \frac{(z/2)^\nu}{\Gamma(\nu+1)}\{\ln(z/2) - \psi(1+\nu)\}, \quad |z| \ll 1.$$

Furthermore, it may be shown that

$$(7) \qquad \frac{\partial I_\nu(z)}{\partial\nu} \cong -\frac{\nu}{\sqrt{2\pi}} z^{-3/2} e^z \quad \text{for } |z| \gg 1 \text{ and } |z| \gg |\nu|.$$

Finally, we note the exact result

$$(8) \qquad \left.\frac{\partial I_\nu(z)}{\partial\nu}\right|_{\nu=0} = -K_0(z)$$

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Gathering together the information in (5)–(8), we then have the following qualitative picture of the behavior of $\partial I_\nu(z)/\partial\nu$:

$$\begin{array}{lccc} & \nu < 0 & \nu = 0 & \nu > 0 \\ z \to 0+ & -\infty & -\infty & 0- \\ z \to \infty & +\infty & 0 & -\infty \end{array} \tag{9}$$

For the specific cases of interest here, we obtain from (5)

$$\left.\frac{\partial I_\nu(z)}{\partial\nu}\right|_{\nu=+1/3}$$

$$= \ln(z/2)I_{1/3}(z) - (z/2)^{1/3}\{-.147\,857\,57 + .519\,020\,49\,(z/2)^2 + .188\,350\,96\,(z/2)^4 + .024\,234\,(z/2)^6 + \cdots\} \tag{10a}$$

$$\left.\frac{\partial I_\nu(z)}{\partial\nu}\right|_{\nu=-1/3}$$

$$= \ln(z/2)I_{-1/3}(z) - (z/2)^{-1/3}\{-.973\,500\,44 + .201\,347\,58\,(z/2)^2 + .259\,796\,06\,(z/2)^4 + .048\,052\,(z/2)^6 + \cdots\}. \tag{10b}$$

These expressions provide a convenient means for evaluating $\partial I_\nu(z)/\partial\nu$, $\nu = \pm\frac{1}{3}$ in the vicinity of the origin. The retention of only those terms explicitly written out in (10a–b) insures an accuracy of four decimal places even for values of z comparable to unity.

3. Construction of the Tables. It was found most convenient to construct the tables of $\partial I_\nu(z)/\partial\nu$, $\nu = \pm\frac{1}{3}$ through numerical differentiation of the existing NBS Tables for $I_\nu(z)$, $\nu = \pm\frac{3}{4}, \pm\frac{2}{3}, \pm\frac{1}{3}, \pm\frac{1}{4}$ (see [4]).

The standard Lagrange interpolation formula for a function $S(\nu, z)$ given at the tabular points $\nu_i (= -\frac{3}{4}, \cdots, +\frac{3}{4})$ may be written as

$$S(\nu, z) = \sum_{\nu_i} \mathfrak{L}(\nu_i, \nu) S(\nu_i, z) \tag{11}$$

where

$$\mathfrak{L}(\nu_i, \nu) = \frac{\prod_{\nu_i} (\nu - \nu_i)}{(\nu - \nu_i)\dfrac{\partial}{\partial\nu}\prod_{\nu_i}(\nu - \nu_i)}.$$

Differentiation then yields

$$\left.\frac{\partial S(\nu, z)}{\partial\nu}\right|_{\nu=\pm1/3} = \sum_{\nu_i} \left.\frac{\partial\mathfrak{L}(\nu_i, \nu)}{\partial\nu}\right|_{\nu=\pm1/3} S(\nu_i, z). \tag{12}$$

In the present instance the detailed computations lead to the expressions

$$\left.\frac{\partial S(\nu, z)}{\partial\nu}\right|_{\nu=+1/3}$$

$$\begin{aligned} = &-.112\,619\,407\,S(-\tfrac{3}{4}, z) + .243\,315\,508\,S(-\tfrac{2}{3}, z) \\ &- 1.5\,S(-\tfrac{1}{3}, z) + 1.800\,865\,801\,S(-\tfrac{1}{4}, z) \\ &- 12.606\,060\,606\,S(\tfrac{1}{4}, z) + 11.737\,362\,637\,S(\tfrac{1}{3}, z) \\ &+ .729\,946\,524\,S(\tfrac{2}{3}, z) - .292\,810\,458\,S(\tfrac{3}{4}, z), \end{aligned} \tag{13a}$$

$$\left.\frac{\partial S(\nu, z)}{\partial \nu}\right|_{\nu=-1/3}$$

$$\begin{aligned} &= .292\,810\,458\, S(-\tfrac{3}{4}, z) - .729\,946\,524\, S(-\tfrac{2}{3}, z) \\ &\quad - 11.737\,362\,637\, S(-\tfrac{1}{3}, z) + 12.606\,060\,606\, S(-\tfrac{1}{4}, z) \\ &\quad - 1.800\,865\,801\, S(\tfrac{1}{4}, z) + 1.5\, S(\tfrac{1}{3}, z) \\ &\quad - .243\,315\,508\, S(\tfrac{2}{3}, z) + .112\,619\,407\, S(\tfrac{3}{4}, z). \end{aligned} \tag{13b}$$

Comparison with (6) indicates that in the interval $.01 \leqq z \leqq 1.00$ the identification

$$S(\nu, z) \to z^{-\nu} I_\nu(z) \tag{14}$$

yields the "smoothest" differentiation, that is, minimal truncation errors. This may also be confirmed by analytical error estimates. Accordingly, in the interval $.01 \leqq z \leqq 1.00$ the tables of $\partial I_\nu(z)/\partial\nu$, $\nu = \pm\frac{1}{3}$ were actually computed from the expressions

$$\left.\frac{\partial I_\nu(z)}{\partial \nu}\right|_{\nu=+1/3} = z^{1/3}\left(\left.\frac{\partial S(\nu, z)}{\partial \nu}\right|_{\nu=+1/3}\right) + \ln(z) I_{1/3}(z); \tag{15a}$$

$$\left.\frac{\partial I_\nu(z)}{\partial \nu}\right|_{\nu=-1/3} = z^{-1/3}\left(\left.\frac{\partial S(\nu, z)}{\partial \nu}\right|_{\nu=-1/3}\right) + \ln(z) I_{-1/3}(z). \tag{15b}$$

The entire sequence (14) $\to$ (13a–b) $\to$ (15a–b) was carried out automatically on an IBM 1620.

The results of this run were spot checked at $z = 1, .3, .11, .01$ against the corresponding values derived from the series (10a–b); the agreement in each case was to better than 4 decimal places.*

In the interval $1.01 \leqq z \leqq 5.00$, the interpolation function was not modified, that is, we used the simple identification

$$S(\nu, z) \to I_\nu(z); \tag{16}$$

the auxiliary computations (14) and (15a–b) could therefore be omitted in this range. The analytical error estimates (Section 4) assure an accuracy of better than $\pm 2 \times 10^{-4}$ in this portion of the tables. The computations were not continued past $z = 5$ because shortly beyond this point the limited capacity of the machine program (8 significant figures) began to introduce truncation errors into the fourth decimal place.

4. Error Bounds. The numerical differentiation (12) is subject to the error [5]

$$\varepsilon(\pm\tfrac{1}{3}, z) = \frac{1}{8!} \prod_{\nu_i}{}' (\pm\tfrac{1}{3} - \nu_i) \left.\frac{\partial^8}{\partial \nu^8} S(\nu, z)\right|_{\nu=\nu_0} \tag{17}$$

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* $\frac{\partial I_\nu(.01)}{\partial \nu}$, $\nu = -\frac{1}{3}$ is an exceptional case since 4D accuracy requires 6 significant figures; this table entry was computed from the series (10b). As a further precaution the entire range $.01 \leqq z \leqq .11$ was checked with the series for $\nu = -\frac{1}{3}$; no further changes were necessary. The basic reason for this difficulty lies in the divergence of $\partial I_\nu(z)/\partial\nu$ for $\nu < 0$ and $z \to 0$ [compare (9)].

where the prime indicates the omission of the zero term, $\nu_i = \pm\frac{1}{3}$; and ν_0 denotes some number lying in the interval $[-\frac{3}{4}, +\frac{3}{4}]$. If we make the identification (16), we find that the upper bound for the error may be put into the form

$$(18)\qquad |\varepsilon(\pm\tfrac{1}{3}, z)| < \frac{5\times 10^{-3}}{8!}\,\mathrm{Max}\left\{\frac{\partial^8}{\partial\nu^8} I_\nu(z)\right\},\quad -\tfrac{3}{4} \leqq \nu \leqq +\tfrac{3}{4}.$$

The problem therefore becomes one of finding majorants for the derivatives of modified Bessel functions. We approach this problem by recalling the identity (Laplace integral)

$$(19)\qquad \frac{(z/2)^{2(m+\nu)}}{\Gamma(\nu+m+1)} = \frac{e^{\tilde{a}(z/2)^2}}{2\pi}\int_{-\infty}^{+\infty} dt\,\frac{e^{i(z/2)^2 t}}{(\tilde{a}+it)^{\nu+m+1}}\,;\ \tilde{a} > 0,\quad \begin{array}{l}\mathrm{Re}\{\nu+m+1\} > 0,\\ \mathrm{Im}\{z\} = 0,\quad z \neq 0;\end{array}$$

which—when substituted into (2) allows its conversion to an integral representation. After the usual (justifiable) interchange of sum and integral we find

$$(20)\qquad I_\nu(z) = \frac{e^{az/2}}{2\pi}\int_{-\infty}^{+\infty} d\tau(a+i\tau)^{-(\nu+1)}\exp\left\{\frac{z}{2}\left[\frac{a}{a^2+\tau^2} + i\tau\left(1-\frac{1}{a^2+\tau^2}\right)\right]\right\};$$

where $a = \tilde{a}z/2$, and in the present instance $a > 0$.

From (20) it follows that

$$(21)\qquad \begin{aligned}\frac{d^N}{\partial\nu^N} I_\nu(z) = \frac{e^{az/2}}{2\pi}\int_{-\infty}^{+\infty} d\tau\,[-\ln(a+i\tau)]^N\,(a+i\,\tau)^{-(\nu+1)}\\ \times\exp\left\{\frac{z}{2}\left[\frac{a}{a^2+\tau^2} + i\tau\left(1-\frac{1}{a^2+\tau^2}\right)\right]\right\}.\end{aligned}$$

This representation seems to be a particularly suitable starting point for the construction of tractable upper bounds.*

We first note that as an immediate consequence of (21) we have the inequality

$$(22)\qquad \left|\frac{\partial^N}{\partial\nu^N} I_\nu(z)\right| < \frac{e^{az/2}}{\pi}\int_0^\infty d\tau\,\frac{|\ln(a+i\tau)|^N}{(a^2+\tau^2)^{\frac{\nu+1}{2}}}\exp\left\{\frac{z}{2}\,\frac{a}{a^2+\tau^2}\right\}.$$

It is convenient to split this integration into two parts:

$$(23)\qquad \int_0^\infty \to \int_0^M + \int_M^\infty \equiv Q_1 + Q_2\,;\quad M > 0.$$

Suppose we consider Q_1 and Q_2 separately. For our purposes it will be sufficient if the integrand of Q_1 is first partially reduced through elementary applications of the mean value theorem. The result may be written

$$(24)\qquad Q_1 < e^{\frac{z}{2a}}\,|\ln(a+iM)|^N\int_0^M d\tau(a^2+\tau^2)^{-\frac{\nu+1}{2}}.$$

With the further overestimate $\int_0^M \to \int_0^\infty$ this may be simplified to the form

$$(25)\qquad Q_1 < e^{\frac{z}{2a}}\,\frac{\sqrt{\pi}}{a^\nu}\,|\ln(a+iM)|^N\,\frac{\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)}\,;\quad \nu > 0.$$

* The special case N = 0 is discussed in [6].

Q_2 may be treated by similar methods. We first remove the exponential factor (mean value theorem) and obtain the bound

$$(26) \qquad Q_2 < \exp\{az/2(M^2 + a^2)\} \int_M^\infty d\tau \frac{|\ln(a + i\tau)|^N}{(a^2 + \tau^2)^{\frac{\nu+1}{2}}}.$$

Now it is convenient to impose the condition

$$(27) \qquad M \gg a.$$

The *leading* term of the integral, therefore, is

$$\int_M^\infty d\tau\, \tau^{-(\nu+1)}(\ln \tau)^N.$$

A simple change of variables then leads to the sequence

$$(28) \quad \int_M^\infty d\tau\, \tau^{-(\nu+1)}(\ln \tau)^N = \frac{1}{\nu^{N+1}} \int_{\nu \ln M}^\infty d\sigma\, \sigma^N e^{-\sigma} < \frac{\Gamma(N+1)}{\nu^{N+1}}, \quad \nu > 0,$$

which yields a concise upper bound for Q_2.

Combining (22)–(28), we finally obtain the desired majorant:

$$(29) \quad \left| \frac{\partial^N}{\partial \nu^N} I_\nu(z) \right| < \frac{e^{a\frac{z}{2}}}{\pi} \left\{ \frac{\sqrt{\pi}}{a^\nu} e^{z/2a} (\ln M)^N \frac{\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)} + e^{az/2M^2} \frac{\Gamma(N+1)}{\nu^{N+1}} \right\},$$

$$M \gg a > 0, \quad N \geqq 0, \qquad \nu > 0, \quad z > 0.$$

For our present applications, where $N = 8$ and $0 < z \leqq 5$, it is convenient to fix a so that $a \approx 1$. Despite (27), it can then evidently be arranged so that

$$(\ln M)^8 \ll \Gamma(9).$$

Under these circumstances it is obvious that the second term on the right hand side of (29) gives the major contribution. If we now combine (18) and (29) we find that the error involved in the numerical differentiation is bounded from above by the expression:

$$(30) \qquad |\varepsilon(\pm \tfrac{1}{3}, z)| \lesssim \frac{5}{\pi} 10^{-3} e^{z/2}/\nu^9.$$

One difficulty remains: In order to be certain of having taken into account the *worst* possible case in (18), ν must be free to range over the entire interval $[-\frac{3}{4}, +\frac{3}{4}]$; on the other hand, it is obvious from (30) that it would be advantageous to have ν as large as possible. Worse yet, it is clear from (29) that the entire mathematical argument is restricted by the condition $\nu > 0$. In order to resolve this difficulty it is necessary to draw on one more property of $I_\mu(z)$: We recall that $I_\mu(z)$, $I_{\mu+1}(z)$, and $I_{\mu+2}(z)$ are connected by the simple recurrence relation

$$(31) \qquad I_\mu(z) = \frac{2(\mu+1)}{z} I_{\mu+1}(z) + I_{\mu+2}(z).$$

By iterating such "step-up" equations it is possible to relate $I_{\mu-1}(z)$, where for example $\mu - 1$ lies in the interval $[-\frac{3}{4}, 0]$, to $I_{\mu+2}(z)$ where the index now ranges over $[2\frac{1}{4}, 3]$. The differential estimates (29) may then advantageously be applied to the Bessel functions of higher index. The detailed relations required in the present context are

$$\begin{aligned}\frac{\partial^8}{\partial\mu^8} I_{\mu-1}(z) &= \frac{8}{z^3}\frac{\partial^8}{\partial\mu^8}\{\mu(\mu+1)(\mu+2)I_{\mu+2}(z)\} \\ &+ \frac{4}{z}\frac{\partial^8}{\partial\mu^8}\{(\mu+1)I_{\mu+2}(z)\} + \frac{4}{z^2}\frac{\partial^8}{\partial\mu^8}\{\mu(\mu+1)I_{\mu+3}(z)\} \\ &+ \frac{\partial^8}{\partial\mu^8} I_{\mu+3}(z); \quad \tfrac{1}{4} \leqq \mu \leqq 1.\end{aligned} \tag{32}$$

Carrying out the differentiations and applying (29) (legitimately now!) to each term on the right hand side one easily finds that the dominant contribution comes from the z^{-3} term. The final error bound is, therefore, given by

$$| \varepsilon(\pm \tfrac{1}{3}, z) | \gtrsim 2 \times 10^{-4}\frac{e^{z/2}}{z^3}. \tag{33}$$

Since the interpolation (13a–b) and series (10a–b) check to better than 4 decimal places at $z = 1$, we are confident that (33) is in fact too pessimistic. The sensitive z-dependence is, of course, the basic reason for resorting to the detour (14) in performing the numerical differentiation for the range $.01 \leqq z \leqq 1.00$.

The recurrence relation (31) as well as the Wronskian

$$\begin{aligned}&\frac{\partial}{\partial\nu}\{I_\nu(z)\}I_{1-\nu}(z) + I_\nu(z)\frac{\partial}{\partial\nu}\{I_{1-\nu}(z)\} \\ &\quad - \frac{\partial}{\partial\nu}\{I_{-\nu}(z)\}I_{\nu-1}(z) - I_{-\nu}(z)\frac{\partial}{\partial\nu}\{I_{\nu-1}(z)\} = -\frac{2}{z}\cos(\pi\nu)\end{aligned} \tag{34}$$

may be used to extend and check the tables.

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z	$\partial I_\nu(z)/\partial\nu$, $\nu = +\frac{1}{3}$	$\partial I_\nu(z)/\partial\nu$, $\nu = -\frac{1}{3}$	z	$\partial I_\nu(z)/\partial\nu$, $\nu = +\frac{1}{3}$	$\partial I_\nu(z)/\partial\nu$, $\nu = -\frac{1}{3}$
0	0	$-\infty$	0.45	−0.9813	−0.3681
0.01	−0.9893	−17.1897	.46	.9754	.3439
.02	−1.0793	−11.2693	.47	.9695	.3208
.03	.1236	−8.6328	.48	.9637	.2986
.04	.1494	−7.0634	.49	.9579	.2772
0.05	−1.1653	−5.9965	0.50	−0.9522	−0.2566
.06	.1752	−5.2128	.51	.9465	.2369
.07	.1810	−4.6071	.52	.9409	.2179
.08	.1840	−4.1218	.53	.9353	.1996
.09	.1849	−3.7223	.54	.9298	.1819
0.10	−1.1842	−3.3866	0.55	−0.9244	−0.1649
.11	.1824	−3.0996	.56	.9190	.1485
.12	.1796	−2.8510	.57	.9137	.1327
.13	.1761	−2.6332	.58	.9084	.1174
.14	.1719	−2.4405	.59	.9032	.1027
0.15	−1.1673	−2.2686	0.60	−0.8980	−0.0884
.16	.1623	−2.1142	.61	.8929	.0747
.17	.1570	−1.9746	.62	.8878	.0614
.18	.1514	.8478	.63	.8828	.0485
.19	.1456	.7319	.64	.8779	.0360
0.20	−1.1396	−1.6257	0.65	−0.8729	−0.0240
.21	.1335	.5278	.66	.8681	.0123
.22	.1273	.4374	.67	.8633	−0.0010
.23	.1210	.3536	.68	.8586	+0.0100
.24	.1146	.2757	.69	.8539	.0206
0.25	−1.1082	−1.2031	0.70	−0.8493	+0.0309
.26	.1017	.1352	.71	.8447	.0409
.27	.0952	.0716	.72	.8402	.0506
.28	.0887	−1.0119	.73	.8357	.0600
.29	.0821	−0.9559	.74	.8312	.0691
0.30	−1.0756	−0.9030	0.75	−0.8269	+0.0779
.31	.0691	.8531	.76	.8226	.0865
.32	.0626	.8060	.77	.8183	.0949
.33	.0561	.7614	.78	.8141	.1030
.34	.0497	.7191	.79	.8099	.1109
0.35	−1.0433	−0.6789	0.80	−0.8058	+0.1186
.36	.0369	.6408	.81	.8017	.1261
.37	.0305	.6045	.82	.7977	.1333
.38	.0242	.5700	.83	.7937	.1404
.39	.0179	.5371	.84	.7898	.1473
0.40	−1.0117	−0.5057	0.85	−0.7859	+0.1540
.41	−1.0056	.4757	.86	.7821	.1605
.42	−0.9994	.4470	.87	.7783	.1668
.43	.9933	.4195	.88	.7745	.1730
.44	.9873	.3933	.89	.7708	.1790

z	$\partial I_\nu(z)/\partial\nu$, $\nu = +\frac{1}{3}$	$\partial I_\nu(z)/\partial\nu$, $\nu = -\frac{1}{3}$	z	$\partial I_\nu(z)/\partial\nu$, $\nu = +\frac{1}{3}$	$\partial I_\nu(z)/\partial\nu$, $\nu = -\frac{1}{3}$
0.90	−0.7672	+0.1849	2.75	−0.6479	+0.5979
.91	.7636	.1906	.80	.6570	.6098
.92	.7600	.1962	.85	.6666	.6222
.93	.7565	.2016	.90	.6769	.6351
.94	.7531	.2069	.95	.6879	.6485
0.95	−0.7497	+0.2121	3.00	−0.6996	+0.6624
.96	.7463	.2171	.05	.7119	.6768
.97	.7429	.2221	.10	.7249	.6919
.98	.7397	.2269	.15	.7387	.7075
.99	.7364	.2316	.20	.7532	.7238
1.00	−0.7332	+0.2362	3.25	−0.7685	+0.7407
.05	.7178	.2576	.30	.7845	.7583
.10	.7034	.2767	.35	.8013	.7766
.15	.6900	.2939	.40	.8190	.7957
.20	.6776	.3095	.45	.8375	.8155
1.25	−0.6661	+0.3237	3.50	−0.8569	+0.8361
.30	.6554	.3366	.55	.8771	.8575
.35	.6456	.3485	.60	.8983	.8798
.40	.6366	.3596	.65	.9205	.9030
.45	.6284	.3699	.70	.9436	.9271
1.50	−0.6210	+0.3796	3.75	−0.9677	+0.9521
.55	.6144	.3888	.80	.9929	.9782
.60	.6085	.3975	.85	−1.0192	+1.0052
.65	.6033	.4059	.90	.0466	.0334
.70	.5987	.4140	.95	.0752	.0627
1.75	−0.5949	+0.4219	4.00	−1.1049	+1.0931
.80	.5917	.4296	.05	.1359	.1248
.85	.5892	.4373	.10	.1682	.1577
.90	.5873	.4449	.15	.2018	.1919
.95	.5861	.4524	.20	.2368	.2274
2.00	−0.5854	+0.4600	4.25	−1.2733	+1.2644
.05	.5854	.4677	.30	.3112	.3028
.10	.5860	.4755	.35	.3507	.3427
.15	.5872	.4834	.40	.3918	.3843
.20	.5890	.4914	.45	.4345	.4274
2.25	−0.5914	+0.4997	4.50	−1.4790	+1.4723
.30	.5943	.5081	.55	.5253	.5189
.35	.5979	.5168	.60	.5734	.5674
.40	.6020	.5258	.65	.6235	.6178
.45	.6068	.5350	.70	.6756	.6702
2.50	−0.6121	+0.5446	4.75	−1.7298	+1.7247
.55	.6181	.5545	.80	.7861	.7813
.60	.6246	.5648	.85	.8447	.8402
.65	.6318	.5754	.90	.9057	.9014
.70	.6395	.5865	.95	−1.9692	+1.9651
			5.00	−2.0351	+2.0312