On the Computation of Euler's Constant

By Dura W. Sweeney

1. Introduction. The computation of Euler's constant, γ, to 3566 decimal places by a procedure not previously used is described. As a part of this computation, the natural logarithm of 2 has been evaluated to 3683 decimal places. A different procedure was used in computations of γ performed by J. C. Adams in 1878 [1] and J. W. Wrench, Jr. in 1952 [2], and recently by D. E. Knuth [3]. This latter procedure is critically compared with that used in the present calculation. The new approximations to γ and ln 2 are reproduced *in extenso* at the end of this paper.

2. Evaluation of γ. A new procedure based upon the expansion of the exponential integral, $-E_i(-x)$, was used to evaluate γ rather than the classical approach used by Adams, Wrench, and Knuth. This new procedure was chosen so as to avoid the more complex programming required in the computation of high orders of Bernoulli numbers.

The exponential integral is given as

$$(1) \qquad -E_i(-x) = \int_x^\infty \frac{e^{-t}\,dt}{t} = -\gamma - \ln x + x - \frac{x^2}{2\cdot 2!} + \frac{x^3}{3\cdot 3!} - \cdots = -\gamma - \ln x + S(x).$$

Its asymptotic expansion is

$$(2) \qquad -E_i(-x) = \int_x^\infty \frac{e^{-t}\,dt}{t} \cong \frac{e^{-x}}{x}\left(1 - \frac{1}{x} + \frac{2!}{x^2} - \cdots\cdot\right) = R(x).$$

Equating these and moving γ to the left, we have

$$(3) \qquad \gamma \cong S(x) - \ln x - R(x).$$

Since the asymptotic form behaves as e^{-x}/x for large x, the difference between $S(x)$ and $\ln x$ will approximate γ to the accuracy of the number of leading zeros in the value of $R(x)$.

$$(4) \qquad \text{For} \quad x = 8192, \quad R(x) = 0.22190 \cdots 10^{-3561}.$$

The value of x was chosen as a power of 2 to simplify the calculation of $\ln x$. Also, since a binary computer was to be used, many of the multiplications in the terms of $S(x)$ could be reduced to shifting operations.

3. Method of Computation. The computation of ln 2 is very rapid and straightforward on a binary computer using one of the forms of the expansion

$$(5) \qquad \ln 2 = \frac{1}{2} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} + \cdots.$$

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The computation of $S(x)$ is also straightforward, but requires substantially more computer time since for $x = 8192$ almost 30,000 terms are required for convergence, and up to three times the number of digits in the final answer are required during intermediate computations to avoid truncation errors and to compensate for the loss of significant figures arising from subtractions.

The computer program was written to do the computation two different ways to establish the accuracy of the analysis, the programming, and the system operation. Two different binary-to-decimal conversion routines were also used, one with each of the computations.

The first part of the computer run used the following procedure. The individual terms of the expansion,

$$13 \ln 2 = 13\left[\frac{5}{1\cdot 1\cdot 2^3} + \frac{11}{2\cdot 3\cdot 2^5} + \frac{17}{3\cdot 5\cdot 2^7} \cdots\cdots\right], \tag{6}$$

were evaluated and summed to form ln 8192. $S(x)$ was evaluated by summing the odd and even terms of the expansion separately to avoid subtractions, thus:

$$\begin{aligned} S(x) &= \left(x + \frac{x^3}{3\cdot 3!} + \cdots\cdots\right) \\ &\quad - \left(\frac{x^2}{2\cdot 2!} + \frac{x^4}{4\cdot 4!} + \cdots\cdots\right) = x + \sum D_{2n+1} - \sum D_{2n}. \end{aligned} \tag{7}$$

The individual terms of these sums were computed from an intermediate value, C_{2n}, as follows:

$$C_{2n} = \frac{x^{2n+1}}{(2n)!} = \frac{x^2}{2n(2n-1)} D_{2n-2}, \quad \text{where} \tag{10}$$

$$D_{2n+1} = \frac{C_{2n}}{(2n+1)^2} \quad \text{and} \quad D_{2n} = \frac{C_{2n}}{2nx}. \tag{11}$$

The second part of the computer run used the following procedure. Ln 8192 was evaluated from the following recursion starting at $n = 12{,}300$:

$$\ln 8192 = 13B_1, \qquad B_n = \frac{1}{2}\left(\frac{1}{n} + B_{n+1}\right). \tag{12}$$

$S(x)$ was evaluated by the following recursion starting at $n = 30{,}000$:

$$S(x) = xA_1, \qquad A_n = 1 - \frac{nx}{(n+1)^2} A_{n+1}. \tag{13}$$

The complete computation was performed on the engineering model of the IBM 7094 in 58 minutes. The first part of the computer run took approximately 20 minutes. The second part took approximately 35 minutes. The remaining time was required for non-overlapped printing and punching of results. The same computation was performed again on an IBM 7090 in 114 minutes as part of the tests of the speed and compatibility of the two systems.

The computed values of ln 2 agreed to 3683 decimal places, and the tabulation is believed accurate to that number of decimals. The value of ln 2 confirms the value calculated by H. S. Uhler [4] to 330 decimal places.

The computed values of γ agreed to the same number of decimal places as ln 2, but the accuracy is limited by the value of x to 3561 decimal places. The value of $R(8192)$ given in (4) was subtracted to give the additional five decimal places shown in parentheses in the tabulation. This value of γ is believed accurate to 3566 decimal places and confirms the value calculated by D. E. Knuth to 1270 decimal places.

4. Comparison of Methods. The operating times reported by Knuth presented an opportunity to compare the two methods to determine which might be more useful in extending the value of γ to greater accuracy. An estimate of the time required shows that if the expansion of the exponential integral had been used it would have been substantially faster than the classical method for the evaluation of γ to 1271 decimal places on the Burroughs 220.

(14) For $x = 3000, \quad R(x) < 10^{-1300}$, and

$$\ln 3000 = \frac{7}{8}\ln 10000 + \frac{1}{2}\ln\left(1 - \frac{1}{10}\right) \tag{15}$$
$$= \frac{7}{8}\ln 10000 - \frac{1}{2}\left(\frac{21}{1\cdot 2\cdot 10^2} + \frac{43}{3\cdot 4\cdot 10^4} + \cdots\cdots\right).$$

Knuth reported a time for the evaluation of ln 10000 of approximately 18 minutes. The additional logarithm would take approximately 4 minutes more.

$S(x)$ would require approximately 10,800 terms for convergence and could probably be most efficiently computed as follows:

$$S(x) = \left(x + \frac{x^3}{3\cdot 3!} + \cdots\cdots\right) - \left(\frac{x^2}{2\cdot 2!} + \frac{x^4}{4\cdot 4!} + \cdots\cdots\right) = \sum D_{2n-1} - \sum D_{2n}, \tag{16}$$

where

$$D_{2n-1} = \frac{(2n-2)x}{(2n-1)^2} D_{2n-2} \quad \text{and} \quad D_{2n} = \frac{(2n-1)x}{(2n)^2} D_{2n-1}. \tag{17}$$

The evaluation of each D_n would require a multiplication loop, a division loop, and a summation loop which would be used to evaluate each storage word of accuracy for each of the terms required for the convergence of $S(x)$. These loop operations would require less than 10 milliseconds for each word of storage. Since there are 10,800 terms, all that is required is an estimate of the accuracy or number of storage words for each term.

The upper curve in Figure 1 shows the value of $r = \log_{10}(3000^n/n\cdot n!)$. For $n = 3000$, r reaches its maximum value of almost 1300. At this value of n, the value of D_n must be known to 2600 decimal places (260 words of storage). To avoid truncation errors, at least 2600 decimal places must be carried for each $n < 3000$. As n becomes larger, the required accuracy decreases, reaching 1300 decimal places at $n \cong 8200$, and going to zero at $n \cong 10{,}800$. This is shown as the difference at a particular n between the upper and lower curves in Figure 1. If the accuracy is carried to 2600 decimal places throughout, as shown by the area between the two curves plus the area outlined by the dotted line, the computation of $S(x)$ would have taken 7.8 hours, i.e.,

$$\left(\frac{260 \times 10{,}800 \times .010}{3600}\right).$$

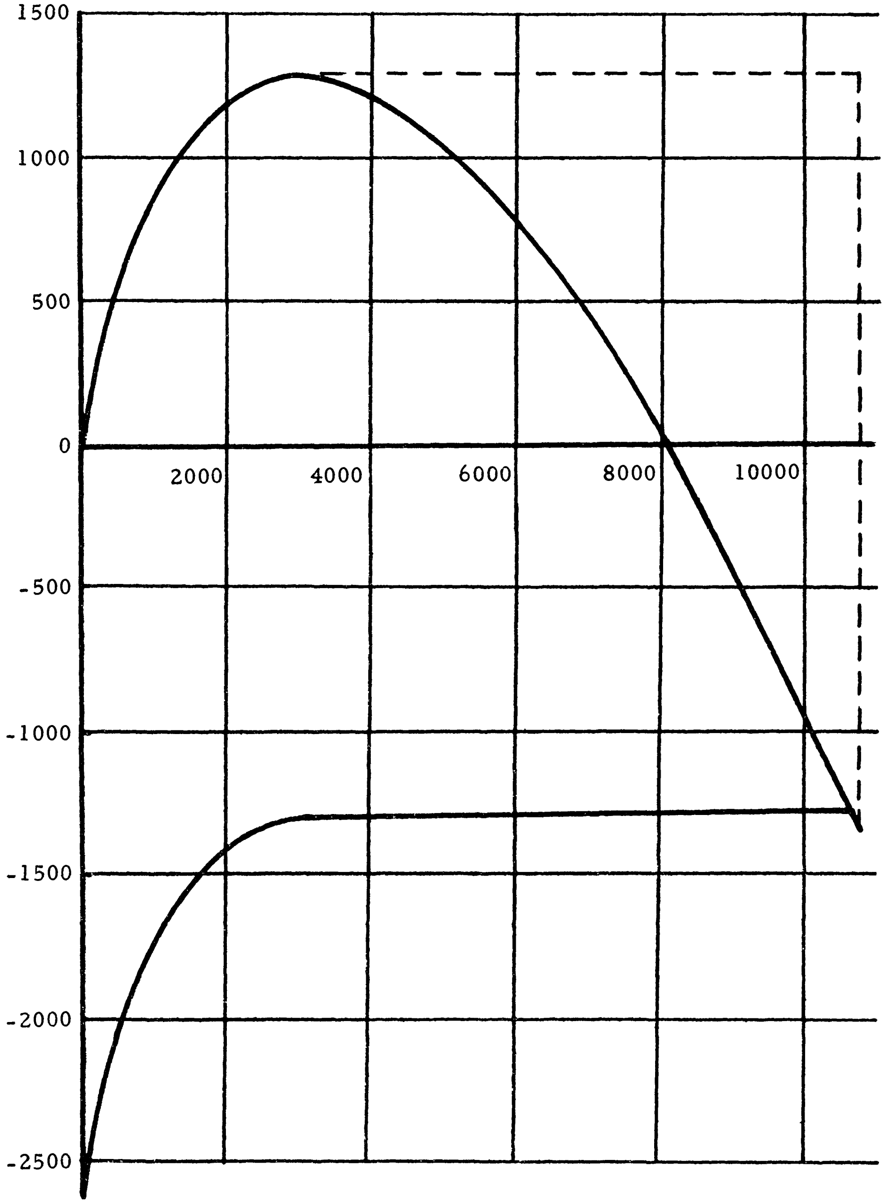

FIG. 1. $r = \log_{10}\left(\frac{3000^n}{n \cdot n!}\right)$

Since the area bounded by the two curves is less than 75% of the total area considered above, an achievable and still faster time for the evaluation of $S(x)$ would be approximately 5.8 hours. This is to be compared to the approximately 9 hours reported by Knuth for the evaluation of the sum of the first 10,000 reciprocals and the first 250 Bernoulli numbers. When the times for the evaluation of the logarithms are added, the comparison shows that the evaluation of γ by this new method would have required about two-thirds of the time reported by Knuth.

A similar comparison was attempted for the evaluation of γ to 3566 decimal places on an IBM 7094 using the classical method. This would have required the evaluation of the sum of the first 65,536 reciprocals and the first 610 Bernoulli numbers. This approach was abandoned since a "good" lower bound of the time required could not be established with reasonable effort because of the complexity in establishing the accuracy (number of words of storage) needed for each of the Bernoulli numbers used in the recursion for evaluating the next higher Bernoulli number. It also appeared that the storage capacity of the system would have been exceeded, requiring additional time and programming complexity. No auxiliary storage is required for the evaluation of γ using the expansion of the exponential integral on either computer.

It should be noted that there exists a still faster method which remains to be tried. This method will require additional programming effort, but substantially less computer time will be required. For a given x evaluate $\ln x$, $S(x)$ and e^{-x} to twice the number of decimal places which would be expected from the value of $R(x)$. Then evaluate the semi-convergent portion of $R(x)$ and multiply by the value of e^{-x}. When this value of $R(x)$ is subtracted from $S(x) - \ln x$, the accuracy of γ will be extended to that expected from the value of $R(2x)$. This method will be faster since $S(x)$ will require far fewer terms for convergence to a certain accuracy than $S(2x)$; e.g., $S(8192)$ will require approximately 36,000 terms for convergence to about 7200 decimal places, while $S(16384)$ will require almost 60,000 terms to achieve the same accuracy.

IBM Data Processing Division
Poughkeepsie, New York

→ J. C. Adams, "On the value of Euler's constant," *Proc. Roy. Soc. London*, v. 27, 1878, p. 88–94.

2. J. W. Wrench, Jr., "A new calculation of Euler's constant," *MTAC*, v. 6, 1952, p. 255.

3. D. E. Knuth, "Euler's constant to 1271 places," *Math. Comp.*, v. 16, 1962, p. 275–281.

→ H. S. Uhler, "Recalculation and extension of the modulus and of the logarithms of 2, 3, 5, 7, and 17," *Proc. Nat. Acad. Sci.*, v. 26, 1940, p. 205–212.

$\gamma =$ .57721 56649 01532 86060 65120 90082 40243 10421 59335 93992
35988 05767 23488 48677 26777 66467 09369 47063 29174 67495
14631 44724 98070 82480 96050 40144 86542 83622 41739 97644
92353 62535 00333 74293 73377 37673 94279 25952 58247 09491
60087 35203 94816 56708 53233 15177 66115 28621 19950 15079

84793 74508 57057 40029 92135 47861 46694 02960 43254 21519
05877 55352 67331 39925 40129 67420 51375 41395 49111 68510
28079 84234 87758 72050 38431 09399 73613 72553 06088 93312
67600 17247 95378 36759 27135 15772 26102 73492 91394 07984
30103 41777 17780 88154 95706 61075 01016 19166 33401 52278

93586 79654 97252 03621 28792 26555 95366 96281 76388 79272
68013 24310 10476 50596 37039 47394 95763 89065 72967 92960
10090 15125 19595 09222 43501 40934 98712 28247 94974 71956
46976 31850 66761 29063 81105 18241 97444 86783 63808 61749
45516 98927 92301 87739 10729 45781 55431 60050 02182 84409

60537 72434 20328 54783 67015 17739 43987 00302 37033 95183
28690 00155 81939 88042 70741 15422 27819 71652 30110 73565
83396 73487 17650 49194 18123 00040 65469 31429 99297 77956
93031 00503 08630 34185 69803 23108 36916 40025 89297 08909
85486 82577 73642 88253 95492 58736 29596 13329 85747 39302

37343 88470 70370 28441 29201 66417 85024 87333 79080 56275
49984 34590 76164 31671 03146 71072 23700 21810 74504 44186
64759 13480 36690 25532 45862 54422 25345 18138 79124 34573
50136 12977 82278 28814 89459 09863 84600 62931 69471 88714
95875 25492 36649 35204 73243 64109 72682 76160 87759 50880

95126 20840 45444 77992 29915 72482 92516 25127 84276 59657
08321 46102 98214 61795 19579 59095 92270 42089 89627 97125
53632 17948 87376 42106 60607 06598 25619 90102 88075 61251
99137 51167 82176 43619 05705 84407 83573 50158 00560 77457
93421 31449 88500 78641 51716 15194 56570 61704 32450 75008

16870 52307 89093 70461 43066 84817 91649 68425 49150 49672
43121 83783 87535 64894 95086 84541 02340 60162 25085 15583
86723 49441 87880 44094 07701 06883 79511 13078 72023 42639
52269 20971 60885 69083 82511 37871 28368 20491 17892 59447
84861 99118 52939 10293 09905 92552 66917 27446 89204 43869

71114 71745 71574 57320 39352 09122 31608 50868 27558 89010
94516 81181 01687 49754 70969 36667 12102 06304 82716 58950
49327 31486 08749 40207 00674 25909 18248 75962 13738 42311
44265 31350 29230 31751 72257 22162 83248 83811 24589 57438
62398 70375 76628 55130 33143 92999 54018 53134 14158 62127

88648 07611 00301 52119 65780 06811 77737 63501 68183 89733
89663 98689 57932 99145 63886 44310 37060 80781 74489 95795
83245 79418 96202 60498 41043 92250 78604 60362 52772 60229
19682 99586 09883 39013 78717 14226 91788 38195 29844 56079
16051 97279 73604 75910 25109 95779 13351 57917 72251 50254

92932 46325 02874 76779 48421 58405 07599 29040 18557 64599
01862 69267 76437 26605 71176 81336 55908 81554 81074 70000
62336 37252 88949 55463 69714 33012 00791 30855 52639 59549
78230 23144 03914 97404 94746 82594 73208 46185 24605 87766
94882 87953 01040 63491 72292 18580 08706 77069 04279 26743

γ (*continued*)

28444	69685	14971	82567	80958	41654	49185	14575	33196	40633
11993	73821	57345	08749	88325	56088	88735	28019	01915	50896
88554	68259	24544	45277	28173	05730	10806	06177	01136	37731
82462	92466	00812	77162	10186	77446	84959	51428	17901	45111
94893	42288	34482	53075	31187	01860	97612	24623	17674	97755
64124	61983	85640	14841	23587	17724	95542	24820	16151	76579
94080	62968	34242	89057	25947	39269	63863	38387	43805	47131
96764	29268	37249	07608	75073	78528	37023	04686	50349	05120
34227	21743	66897	92848	62972	90889	26789	77703	26246	23912
26188	87653	00577	86274	36060	94443	60392	80977	08133	83693
42355	08583	94112	67092	18734	41451	21878	03276	15050	94780
55466	30058	68455	63152	45460	53151	13252	81889	10792	31491
31103	23443	02450	93345	00030	76558	64874	22297	17700	33178
45391	50566	94015	99884	92916	09114	00294	86902	08848	53816
97009	55156	63470	55445	22176	40358	62939	82865	81312	38701
32535	88006	25686	62692	69977	67737	73068	32269	00916	08510
45150	02261	07180	25546	59284	93894	92775	95897	54076	15599
33782	64824	19795	06418	68143	78817	18508	85408	03679	96314
23954	00919	64388	75007	89000	00627	99794	28098	86372	99259
19777	65040	40992	20379	40427	61681	78371	56686	53066	93983
09165	24322	70595	53041	76673	66401	16792	95901	29305	37449
71830	80042	7(58486)							

ln 2 = .69314 71805 59945 30941 72321 21458 17656 80755 00134 36025
52541 20680 00949 33936 21969 69471 56058 63326 99641 86875
42001 48102 05706 85733 68552 02357 58130 55703 26707 51635
07596 19307 27570 82837 14351 90307 03862 38916 73471 12335
01153 64497 95523 91204 75172 68157 49320 65155 52473 41395

25882 95045 30070 95326 36664 26541 04239 15781 49520 43740
43038 55008 01944 17064 16715 18644 71283 99681 71784 54695
70262 71631 06454 61502 57207 40248 16377 73389 63855 06952
60668 34113 72738 73722 92895 64935 47025 76265 20988 59693
20196 50585 54764 70330 67936 54432 54763 27449 51250 40606

94381 47104 68994 65062 20167 72042 45245 29612 68794 65461
93165 17468 13926 72504 10380 25462 59656 86914 41928 71608
29380 31727 14367 78265 48775 66485 08567 40776 48451 46443
99404 61422 60319 30967 35402 57444 60703 08096 08504 74866
38523 13818 16767 51438 66747 66478 90881 43714 19854 94231

51997 35488 03751 65861 27535 29166 10007 10535 58249 87941
47295 09293 11389 71559 98205 65439 28717 00072 18085 76102
52368 89213 24497 13893 20378 43935 30887 74825 97017 15591
07088 23683 62758 98425 89185 35302 43634 21436 70611 89236
78919 23723 14672 32172 05340 16492 56872 74778 23445 35347

64811 49418 64238 67767 74406 06956 26573 79600 86707 62571
99184 73402 26514 62837 90488 30620 33061 14463 00737 19489
00274 36439 65002 58093 65194 43041 19115 06080 94879 30678
65158 87090 06052 03468 42973 61938 41289 65255 65396 86022
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91589 42665 47859 59648 90653 05846 02586 68382 94002 28330
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72430 02489 79454 01961 87233 98086 08316 64811 49093 06675

19339 31289 04316 41370 68139 77764 98176 97486 89038 87789
99129 65036 19270 71088 92641 05230 92478 39173 73501 22984
24204 99568 93599 22066 02204 65494 15106 13918 78857 44245
57751 02068 37030 86661 94808 96412 18680 77902 08181 58858
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20602 53959 54365 41655 31129 51759 89940 05600 03665 13567
56905 12459 26825 74394 64831 68332 62490 18038 24240 82423
14523 06140 96380 57007 02551 38770 26817 85163 06902 55137
03234 05380 21450 19015 37402 95099 42262 99577 96474 27138
15736 38017 29873 94070 42421 79972 26696 29799 39312 70693

57472 40493 38653 08797 58721 69964 51294 46491 88377 11567
01678 59880 49818 38896 78413 49383 14014 07316 64727 65327
63591 92335 11233 38933 87095 13209 05927 21854 71328 97547
07978 91384 44546 66761 92702 88553 34234 29899 32180 37691
54973 34026 75467 58873 23677 83429 16191 81043 01160 91695

26554 78597 32891 76354 55567 42863 87746 39871 01912 43175
42558 88301 20677 92102 80341 20687 97591 43081 28330 72303
00883 49470 57924 96591 00586 00123 41561 75741 32724 65943
06843 54652 11135 02154 43415 39955 38185 65227 50221 42456
64400 06276 18330 32064 72725 72197 51529 08278 56842 13207

ln 2 (*continued*)

95988	63896	72771	19552	21881	90466	03957	00977	47065	12619
50527	89322	96088	93140	56254	33442	55239	20620	30343	94177
73579	45592	12590	19925	59114	84402	42390	12554	25900	31295
37051	92206	15064	34583	78787	30020	35414	42178	57580	13236
45166	07099	14383	14500	49858	96688	57722	21486	52882	16941
81270	48860	75897	22032	16663	12837	83291	56763	07498	72985
74638	92826	93735	09840	77804	93950	04933	99876	26475	50703
16221	61390	34845	29942	49172	48373	40613	66226	38349	36811
16841	67056	92521	47513	83930	63845	53718	62687	79732	88955
58871	63442	97562	44755	39236	63694	88877	82389	01749	81027
35655	24050	51854	77306	19440	52423	22125	59024	83308	27788
88890	59629	11972	99545	74415	62451	24859	26831	12607	46797
28163	80902	50005	65599	91461	28332	54358	11140	48482	06064
08242	24792	40385	57647	62350	31100	32425	97091	42501	11461
55848	30670	01258	31821	91534	72074	74111	94009	83557	32728
26144	27382	13970	70477	95625	96705	79023	03384	80617	13455
55368	55375	81065	74973	44479	22511	19654	61618	27896	01006
85129	65395	47965	86637	83522	47362	45460	93585	03605	06784
14391	14452	31457	78033	59179	21127	95570	50555	54514	38788
81881	53519	48593	44672	46429	49864	05062	65184	24475	39566
37833	73482	20753	32944	81306	49336	03546	10101	77464	93267
87716	71986	12073	96832	01235	96077	29024	68304	59403	13056
37763	13240	10804	20285	43590	26945	09403	07400	14933	95076
73160	28502	86973	03187	18239	98433	525			