

# Fast Method for Computing the Number of Primes Less than a Given Limit

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**Abstract.** "Fast Method for Computing the Number of Primes Less Than a Given Limit" describes three processes used during the course of calculation. In the first part of the paper the author proves:

$$\phi(x, a) = \phi(x, 1) - \phi\left(\frac{x}{p_2}, 1\right) - \phi\left(\frac{x}{p_3}, 2\right) - \dots - \phi\left(\frac{x}{p_a}, a - 1\right)$$

where  $\phi(x, a)$  represents the number of numbers less than or equal to  $x$  and not divisible by the first " $a$ " primes. This identity is used to evaluate the formula  $\pi(x) = \phi(x, a) + a - 1$ ,  $a + 1 > \pi(\sqrt{x})$  where resulting terms of the form  $\phi(x', a')$  are broken down still further by the previously described method, or numerically evaluated using one or both of two other identities, the choice being dependent on  $x'$  and  $a'$ .

Following the paper is a table of calculations made using this process which gives the values of  $\pi(x)$  for  $x$  at intervals of 10 million up to 1000 million, along with the Riemann and the Chebyshev approximations for  $\pi(x)$  and the amount they deviate from the true count.

## 1. Definitions and Notations.

- $\pi(x)$ : the number of primes less than or equal to  $x$ .
- $p_a$ : the  $a$ th prime, ( $p_1 = 2$ ).
- $m_a$ :  $p_1 p_2 \cdots p_a$ .
- $[x]$ : greatest integer  $\leq x$ .
- $c$ : any integer.
- $c \mid x$ :  $c$  divides  $x$ .
- $c \nmid x$ :  $c$  does not divide  $x$ .
- $S(x, a)$ : set of all numbers  $\leq x$  and prime to  $m_a$ .
- $\phi(x, a)$ : number of members of  $S(x, a)$ .
- $\phi(-x, a)$ : equivalent to  $-\phi(x, a)$ .
- $P_s(x, a)$ : number of products of  $s$  primes belonging to  $S(x, a)$ ,  $P_0(x, a) = 1$ .
- $P_s(-x, a)$ : equivalent to  $-P_s(x, a)$ .
- $P_1(x, a)$ : by definition is  $\pi(x) - a$ .

Legendre Sum: 
$$\phi(x, a) = [x] - \sum_{i \leq a} [x/p_i] + \sum_{j < i \leq a} [x/p_i p_j] - \sum_{k < j < i \leq a} [x/p_i p_j p_k] + \dots$$

## 2. Formula Development. Let:

$$(1) \quad T_k(x, a) \equiv (-1)^{\beta_0 + \beta_1 + \dots + \beta_{a-1}} [x/p_1^{\beta_0} p_2^{\beta_1} \cdots p_a^{\beta_{a-1}}],$$

where  $k = 2^{a-1}\beta_{a-1} + 2^{a-2}\beta_{a-2} + \dots + 2^0\beta_0$  such that the  $\beta$ 's are either 1 or 0, thus

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being the digits of  $k$  expressed in binary form and where  $0 \leq k < 2^a$ . Let us also adopt:  $T_k(-x, a) \equiv -T_k(x, a)$ . Legendre's sum may be written in the following order:

$$(2) \quad \phi(x, a) = \sum_{k=0}^{2^a-1} T_k(x, a).$$

Let  $M$  be an integer  $< 2^a$  and let  $2^i$  be the highest power of  $2$  dividing  $M$ . Let

$$(3) \quad \gamma(M, x, a) = \sum_{k=M}^{M+2^i-1} T_k(x, a).$$

Full understanding of (3) should be clear from the identity:

$$(4) \quad \phi(x, a) = T_0(x, a) + \gamma(2^0, x, a) + \gamma(2^1, x, a) + \cdots + \gamma(2^{a-1}, x, a),$$

which is (2) with (3) applied.

From (1) it follows that:

$$(5) \quad \{\text{sgn } T_k(x, a)\}_{k>0, 2^i|k} = (-1)^{\beta_i + \beta_{i+1} + \cdots + \beta_{a-1}}$$

and

$$(6) \quad |T_k(x, a)|_{k>0, 2^i|k} = [x/p_{i+1}^{\beta_i} p_{i+2}^{\beta_{i+1}} \cdots p_a^{\beta_{a-1}}].$$

By substituting  $T_k(x, a)$  for  $x$  and  $i$  for  $a$  in (1) we get:

$$(7) \quad T_{k'}\{T_k(x, a), i\} = \text{sgn}\{T_k(x, a)\}(-1)^{\beta_0' + \beta_1' + \cdots + \beta_{i-1}'} \times [|T_k(x, a)| / p_1^{\beta_0'} p_2^{\beta_1'} \cdots p_i^{\beta_{i-1}'}],$$

where  $k' = 2^{i-1}\beta'_{i-1} + 2^{i-2}\beta'_{i-2} + \cdots + 2^0\beta'_0$  and  $0 \leq k'$ . When  $2^i | k$ , we can substitute (5) and (6) into (7) giving:

$$(8) \quad T_{k'}\{T_k(x, a), i\}_{2^i|k} = T_{k+k'}(x, a).$$

By substituting  $T_M(x, a)$  for  $x$  and  $i$  for  $a$  in (2) we get

$$(9) \quad \phi\{T_M(x, a), i\} = \sum_{k'=0}^{2^i-1} T_{k'}\{T_M(x, a), i\}.$$

Using (8), (9) becomes

$$\phi\{T_M(x, a), i\}_{2^i|M} = \sum_{k=M}^{M+2^i-1} T_k(x, a)$$

and using (3) we have

$$(10) \quad \gamma(M, x, a) = \phi\{T_M(x, a), i\}.$$

Now (4) becomes

$$(11) \quad \phi(x, a) = T_0(x, a) + \phi\{T_{2^0}(x, a), 0\} + \phi\{T_{2^1}(x, a), 1\} + \cdots + \phi\{T_{2^{a-1}}(x, a), a-1\}.$$

By replacing  $x$  with  $T_M(x, a)$  and  $a$  with  $i$ , we get

$$(12) \quad \phi\{T_M(x, a), i\} = T_0\{T_M(x, a), i\} + \phi(T_{2^0}\{T_M(x, a), i\}, 0) + \phi(T_{2^1}\{T_M(x, a), i\}, 1) + \cdots + \phi(T_{2^{i-1}}\{T_M(x, a), i\}, i-1).$$

Using (1) to find  $T_0\{T_M(x, a), i\}$  and (8) in general, we get

$$(13) \quad \phi\{T_M(x, a), i\}_{2^i|M} = T_M(x, a) + \phi\{T_{M+2^0}(x, a), 0\} \\ + \phi\{T_{M+2^1}(x, a), 1\} + \cdots + \phi\{T_{M+2^{i-1}}(x, a), i - 1\}.$$

We can now calculate  $\phi(x, a)$  by (13) setting  $\phi(x, a)$  equal to  $\phi\{T_0(x, a), a\}$  where  $2^a | 0$ . In (13)  $T_M(x, a)$  is computed by (1) while the remaining terms are computed by reapplication of (13) or, if reduced by (1) to a numerical expression of  $\phi(x, a)$ , by any other method of computing  $\phi(x, a)$ .

**3. Other Methods of Calculating  $\phi(x, a)$ .** The following method for calculating  $\phi(x, a)$  was developed by D. H. Lehmer [1]. Meissel's formula for finding  $\pi(x)$  is a special case of this method [1].

$$(14) \quad \phi(x, a) = \sum_{s=0}^{V-1} P_s(x, a), \quad x < p_{a+1}^V$$

where

$$P_0(x, a) = 1, \\ P_1(x, a) = \pi(x) - a, \\ P_2(x, a) = \sum_{p_a < p_i \leq x/p_i} \{\pi(x/p_i) - (i - 1)\}, \\ P_3(x, a) = \sum_{p_a < p_i \leq x/p_i^2} \sum_{p_i \leq p_j \leq x/p_i p_j} \{\pi(x/p_i p_j) - (j - 1)\}.$$

If  $V - 1$  is greater than 3 this method of calculating  $\phi(x, a)$  may not be practical. Also

$$(15) \quad \phi(x, a) = c\phi(m_a, a) + \phi(r, a)$$

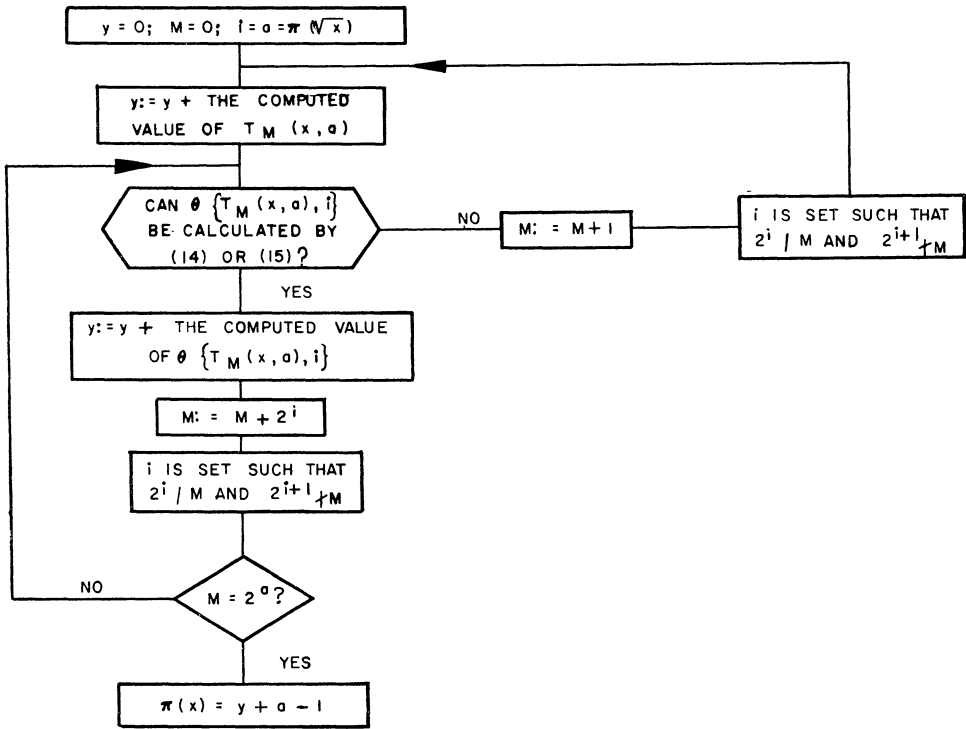
where  $x = cm_a + r$ ,  $|r| \leq \frac{1}{2}m_a$ , and  $a > 1$ . Hence  $c\phi(m_a, a)$  can be calculated using a table of the values for  $\phi(m_a, a)$  and  $\phi(r, a)$  by (14) or (13).

**4. Fast Method for Computing  $\pi(x)$ .** From the above development we can formulate the following procedure for computing  $\pi(x)$ . Use the formula:

$$(16) \quad \pi(x) = \phi(x, a) + a - 1$$

where  $p_{a+1}^2 \geq x \geq p_a^2$ . Place a limit on  $V$  in (14) such that  $\phi(x, a)$  is not computable by (14) when  $p_{a+1}^4 \leq x$ . Now place a limit on  $a$  in (15) such that  $\phi(r, a)$  will always be computable by (14) with its restriction  $V \leq 4$ . This means the value of  $a$  in (15) must meet the requirement  $1 < a \leq 6$ . With a limited table of primes, the larger the better, we are prepared to compute  $\pi(x)$ .

Compute  $\phi(x, a)$  in (16) by (13) where the resulting term  $T_0(x, a)$  is computed by (1). The rest of the resulting terms from (13) considered as expressions of the form  $\phi(x, a)$  are computed by (1) if  $a = 1$ , by (15),  $\phi(r, a)$  being computed by (14), if  $1 < a \leq 6$ , and by (14) or the reapplication of (13) if  $a > 6$ . In the latter case  $\phi(x, a)$  is computed by (14) unless  $V > 4$  or  $\pi(x)$  is too large to be found in the table of primes when  $2 \leq V \leq 4$ . The resulting terms from a reapplication of (13) are calculated using the process used in calculating the resulting terms of the calculation of  $\phi(x, a)$  in (16).



We now introduce a system of order as an aid to our method of calculating  $\pi(x)$ . At any point in our calculations we will have a number of terms calculated. Introducing  $y$  as the sum of these calculated terms, here is the system of order in flow chart form.

The length of time necessary for the calculation of  $\pi(x)$  on the 709 using the author's program was:

$$t \sim 60x^{\log_{10} 5} \mu\text{sec.}$$

In the following table the "Li" function is Chebyshev's approximation for  $\pi(x)$  and the "R" function is the Riemann approximation

$$Li(x) = \int_2^x \frac{dt}{\ln t},$$

$$R(x) = \sum_{n=1}^{\infty} n^{-1} \mu(n) Li(x^{1/n}).$$

They are computed using

$$Li(x) = \gamma + \ln(\ln x) + \ln x + \frac{(\ln x)^2}{2 \times 2!} + \frac{(\ln x)^3}{3 \times 3!} + \dots + \frac{(\ln x)^{75}}{75 \times 75!},$$

$$R(x) = 1 + \frac{\ln x}{S_2} + \frac{(\ln x)^2}{S_3(2 \times 2!)} + \dots + \frac{(\ln x)^{75}}{S_{76}(75 \times 75!)}$$

where  $S_n = \sum_{k=1}^{\infty} k^{-n}$ .

Although the table gives  $x$  only at intervals of 10 million, calculations have also been made for  $x$  at intervals of 1 million, with the results listed in the same manner. Values of  $\pi(x)$  for  $x$  (in millions) at 1, 2, . . . , 10 checked with values obtained by D. N. Lehmer [2]. The values of  $\pi(x)$  for the following values of  $x$  (in millions) checked with those listed by D. H. Lehmer [1, p. 386]: 20, 25, 33, 37, 40, 90, 100, 999, and 1000.

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1. D. H. LEHMER, *Illinois J. Math.*, v. 3, no. 3, p. 381-388, 1959.
2. D. N. LEHMER, *List of Prime Numbers from 1 to 10,006,721*, New York, Hafner Pub. Co., 1956.

*Comparison of the Count of Primes with the Corresponding Values of the Formulas of Chebyshev and Riemann at intervals of 10,000,000 up to 1,000,000,000.*

$x$	$\pi(x)$	$Li(x)$	$R(x)$	$Li(x) - \pi(x)$	$R(x) - \pi(x)$
10000000	664579	664918	664667	+339	+88
20000000	1270607	1270905	1270571	+298	-36
30000000	1857859	1858213	1857818	+354	-41
40000000	2433654	2434016	2433570	+362	-84
50000000	3001134	3001557	3001067	+423	-67
60000000	3562115	3562683	3562154	+568	+39
70000000	4118064	4118585	4118020	+521	-44
80000000	4669382	4670091	4669493	+709	+111
90000000	5216954	5217810	5217182	+856	+228
100000000	5761455	5762209	5761552	+754	+97
110000000	6303309	6303656	6302971	+347	-338
120000000	6841648	6842446	6841735	+798	+87
130000000	7378187	7378825	7378090	+638	-97
140000000	7912199	7912998	7912239	+799	+40
150000000	8444396	8445139	8444357	+743	-39
160000000	8974458	8975397	8974593	+939	+135
170000000	9503083	9503902	9503077	+819	-6
180000000	10030385	10030768	10029923	+383	-462
190000000	10555473	10556096	10555231	+623	-242
200000000	11078937	11079975	11079090	+1038	+153
210000000	11601626	11602484	11601581	+858	-45
220000000	12122540	12123696	12122775	+1156	+235
230000000	12642573	12643676	12642736	+1103	+163
240000000	13161544	13162482	13161525	+938	-19
250000000	13679318	13680169	13679195	+851	-123
260000000	14195860	14196786	14195796	+926	-64
270000000	14711384	14712378	14711372	+994	-12
280000000	15226069	15226988	15225965	+919	-104
290000000	15739663	15740653	15739614	+990	-49
300000000	16252325	16253409	16252355	+1084	+30
310000000	16764521	16765291	16764222	+770	-299
320000000	17275206	17276328	17275245	+1122	+39
330000000	17785475	17786551	17785453	+1076	-22
340000000	18294605	18295985	18294873	+1380	+268
350000000	18803526	18804658	18803531	+1132	+5
360000000	19311288	19312592	19311452	+1304	+164
370000000	19818405	19819810	19818656	+1405	+251
380000000	20325373	20326334	20325167	+961	-206
390000000	20831210	20832184	20831003	+974	-207
400000000	21336326	21337378	21336185	+1052	-141
410000000	21840713	21841935	21840729	+1222	+16
420000000	22344479	22345872	22344653	+1393	+174
430000000	22848050	22849204	22847973	+1154	-77
440000000	23350555	23351948	23350705	+1393	+150
450000000	23853038	23854119	23852863	+1081	-175
460000000	24354548	24355729	24354461	+1181	-87
470000000	24855718	24856793	24855513	+1075	-205
480000000	25356424	25357324	25356032	+900	-392
490000000	25856368	25857333	25856029	+965	-339
500000000	26355867	26356832	26355517	+965	-350

$x$	$\pi(x)$	$Li(x)$	$R(x)$	$Li(x) - \pi(x)$	$R(x) - \pi(x)$
510000000	26854252	26855833	26854507	+1581	+255
520000000	27352687	27354346	27353008	+1659	+321
530000000	27850698	27852381	27851033	+1683	+335
540000000	28348381	28349949	28348589	+1568	+208
550000000	28845356	28847059	28845688	+1703	+332
560000000	29342150	29343720	29342338	+1570	+188
570000000	29838286	29839940	29838548	+1654	+262
580000000	30334175	30335730	30334327	+1555	+152
590000000	30829544	30831095	30829682	+1551	+138
600000000	31324703	31326045	31324622	+1342	-81
610000000	31819444	31820587	31819153	+1143	-291
620000000	32313388	32314729	32313285	+1341	-103
630000000	32807229	32808477	32807023	+1248	-206
640000000	33300450	33301838	33300374	+1388	-76
650000000	33793395	33794819	33793345	+1424	-50
660000000	34286170	34287427	34285943	+1257	-227
670000000	34778319	34779667	34778173	+1348	-146
680000000	35270167	35271546	35270042	+1379	-125
690000000	35761747	35763069	35761556	+1322	-191
700000000	36252931	36254242	36252719	+1311	-212
710000000	36743905	36745071	36743539	+1166	-366
720000000	37234048	37235561	37234019	+1513	-29
730000000	37724170	37725717	37724166	+1547	-4
740000000	38213987	38215544	38213984	+1557	-3
750000000	38703181	38705046	38703477	+1865	+296
760000000	39192219	39194230	39192652	+2011	+433
770000000	39680979	39683099	39681512	+2120	+533
780000000	40169476	40171658	40170062	+2182	+586
790000000	40658253	40659911	40658306	+1658	+53
800000000	41146179	41147862	41146248	+1683	+69
810000000	41634187	41635516	41633893	+1329	-294
820000000	42121502	42122877	42121245	+1375	-257
830000000	42608404	42609948	42608308	+1544	-96
840000000	43095410	43096733	43095084	+1323	-326
850000000	43581966	43583236	43581579	+1270	-387
860000000	44067840	44069462	44067796	+1622	-44
870000000	44553888	44555412	44553738	+1524	-150
880000000	45039361	45041091	45039408	+1730	+47
890000000	45524412	45526502	45524811	+2090	+399
900000000	46009215	46011649	46009949	+2434	+734
910000000	46494557	46496534	46494826	+1977	+269
920000000	46979583	46981161	46979445	+1578	-138
930000000	47463433	47465532	47463809	+2099	+376
940000000	47947424	47949652	47947920	+2228	+496
950000000	48431471	48433523	48431783	+2052	+312
960000000	48915316	48917147	48915399	+1831	+83
970000000	49398798	49400527	49398771	+1729	-27
980000000	49881580	49883667	49881903	+2087	+323
990000000	50364709	50366569	50364797	+1860	+88
1000000000	50847534	50849235	50847455	+1701	-79