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Supplementary Data and Remarks Concerning a Hardy-Littlewood Conjecture

By Daniel Shanks

Let $P_a(N)$ be the number of primes of the form $n^2 + a$ for $1 \leqq n \leqq N$, and let $\bar{\pi}_a(N)$ be the number of primes $\leqq N$ for which $-a$ is a quadratic nonresidue. In [1] we discussed a conjecture of Hardy and Littlewood to the effect that

$$\frac{P_a(N)}{\bar{\pi}_a(N)} \sim h_a \tag{1}$$

where the constant h_a is given by

$$h_a = \prod_{p \nmid a} \left(1 - \left(\frac{-a}{p}\right) \frac{1}{p-1}\right), \tag{2}$$

the product being taken over the odd primes p, with $(-a/p)$ the Legendre Symbol. We gave in [1] a heuristic argument in support of (1), a method of computing the h_a, and supporting empirical data for the six cases $a = 1, \pm 2, \pm 3$, and 4.

Subsequently the primes were also counted for six other cases, namely $a = \pm 5, \pm 6, \pm 7$, and since such data are not available elsewhere it seems desirable to record them in a brief note. In Tables 1, 2, and 3 we show summaries for $N =$ 10,000 (10,000) 180,000 in the same format as the tables in [1].

While accurate values of h_a in these six cases had not been computed, it was at once apparent that (1) is at least roughly correct for these values of a also. Quite recently [2] tables of $L_a(s)$ for $a = \pm 6$ have been computed by J. W. Wrench, Jr., and, on the basis of these, one finds

$$\begin{aligned} h_6 &= 0.71304162 \\ h_{-6} &= 1.03575587. \end{aligned} \tag{3}$$

These are in good agreement with the empirical ratios in Table 2. Equally accurate constants for $a = \pm 5$ and ± 7 are more difficult to compute, and are not yet available.

We may note the following:

1. Of the twelve forms, $n^2 + a$, that we have investigated, $n^2 + 7$ has the most primes. Its (empirical) h_7, equal to 1.98, indicates that numbers of this form are primes nearly twice as often as numbers of the same magnitude chosen at random.

Received November 8, 1962.

2. On the other extreme, $n^2 + 5$ is a prime only about one-half as often ($h_5 = 0.53$) as numbers of the same magnitude chosen at random.

3. And, of all twelve cases, $n^2 - 6$ is most nearly normal in number of primes, since h_{-6} is the closest to 1.

4. We also note that $n^2 + 6$ and $n^2 + 2$ have constants that are re-

TABLE 1

N	$P_5(N)$	$\bar{\pi}_5(N)$	$P_5(N)/\bar{\pi}_5(N)$	$P_{-5}(N)$	$\bar{\pi}_{-5}(N)$	$P_{-5}(N)/\bar{\pi}_{-5}(N)$
10,000	339	613	0.5530	1088	618	1.7605
20,000	627	1136	0.5519	2051	1138	1.8023
30,000	880	1622	0.5425	2916	1633	1.7857
40,000	1123	2107	0.5330	3780	2112	1.7898
50,000	1376	2589	0.5315	4593	2578	1.7816
60,000	1606	3054	0.5259	5420	3038	1.7841
70,000	1846	3500	0.5274	6214	3479	1.7861
80,000	2099	3945	0.5321	7018	3932	1.7848
90,000	2332	4389	0.5313	7834	4367	1.7939
100,000	2567	4817	0.5329	8579	4813	1.7825
110,000	2802	5238	0.5349	9344	5257	1.7774
120,000	3028	5666	0.5344	10119	5671	1.7843
130,000	3260	6090	0.5353	10858	6105	1.7785
140,000	3493	6519	0.5358	11603	6524	1.7785
150,000	3723	6954	0.5354	12341	6933	1.7800
160,000	3936	7371	0.5340	13097	7350	1.7819
170,000	4148	7763	0.5343	13844	7757	1.7847
180,000	4368	8170	0.5346	14575	8182	1.7813

TABLE 2

N	$P_6(N)$	$\bar{\pi}_6(N)$	$P_6(N)/\bar{\pi}_6(N)$	$P_{-6}(N)$	$\bar{\pi}_{-6}(N)$	$P_{-6}(N)/\bar{\pi}_{-6}(N)$
10,000	444	616	0.7208	643	620	1.0371
20,000	782	1147	0.6818	1155	1142	1.0114
30,000	1147	1633	0.7024	1684	1635	1.0300
40,000	1500	2125	0.7059	2164	2111	1.0251
50,000	1834	2583	0.7100	2649	2565	1.0327
60,000	2157	3049	0.7075	3134	3044	1.0296
70,000	2488	3490	0.7129	3607	3476	1.0377
80,000	2793	3919	0.7127	4086	3913	1.0442
90,000	3123	4352	0.7176	4559	4363	1.0449
100,000	3420	4795	0.7132	5010	4804	1.0429
110,000	3733	5226	0.7143	5462	5238	1.0428
120,000	4038	5650	0.7147	5913	5668	1.0432
130,000	4352	6077	0.7161	6362	6097	1.0435
140,000	4671	6516	0.7169	6801	6530	1.0415
150,000	4978	6937	0.7176	7229	6953	1.0397
160,000	5286	7346	0.7196	7656	7382	1.0371
170,000	5580	7752	0.7198	8098	7793	1.0391
180,000	5889	8160	0.7217	8552	8209	1.0418

TABLE 3

N	$P_7(N)$	$\bar{\pi}_7(N)$	$P_7(N)/\bar{\pi}_7(N)$	$P_{-7}(N)$	$\bar{\pi}_{-7}(N)$	$P_{-7}(N)/\bar{\pi}_{-7}(N)$
10,000	1238	620	1.9968	440	616	0.7143
20,000	2254	1134	1.9877	841	1145	0.7345
30,000	3225	1638	1.9689	1226	1628	0.7531
40,000	4176	2110	1.9792	1590	2112	0.7528
50,000	5094	2569	1.9829	1937	2563	0.7558
60,000	6004	3035	1.9783	2272	3026	0.7508
70,000	6891	3481	1.9796	2617	3479	0.7522
80,000	7788	3948	1.9726	2958	3944	0.7500
90,000	8697	4372	1.9893	3304	4389	0.7528
100,000	9521	4813	1.9782	3627	4828	0.7512
110,000	10419	5250	1.9846	3977	5252	0.7572
120,000	11228	5669	1.9806	4291	5675	0.7561
130,000	12070	6101	1.9784	4632	6109	0.7582
140,000	12904	6523	1.9782	4953	6521	0.7596
150,000	13739	6943	1.9788	5258	6943	0.7573
160,000	14580	7360	1.9810	5585	7362	0.7586
170,000	15450	7765	1.9897	5914	7770	0.7611
180,000	16240	8198	1.9810	6225	8211	0.7581

markably close:

$$h_6 = 0.71304162$$

$$h_2 = 0.71306310.$$

Corresponding to 100,000 primes of the former type, there should be 100,003 primes of the latter. Up to $N = 180{,}000$ the two classes take turns being in the lead, but, if (1) is true, $n^2 + 2$ must eventually take, and hold, the lead. A reasonable estimate, however, suggests that this could be postponed until N exceeds $3 \cdot 10^{10}$.

5. We show in Figure 1 a bar graph of h_a for $a = -20\ (1)\ 9$. These include the previously computed values; the present computed values in equation (3) above and Table 4 below; the empirical values in comments 1 and 2 above; and the easily computed $h_{-9} = h_{-16} = 0$, $h_8 = h_2$, $h_{-8} = h_{-2}$, $h_{-12} = h_{-3}$, $h_{-20} = h_{-5}$, $h_9 = \frac{2}{3} h_1$, and $h_{-18} = \frac{2}{3} h_{-2}$. One sees at a glance that the distribution of primality in the neighborhood of square numbers is anything but uniform.

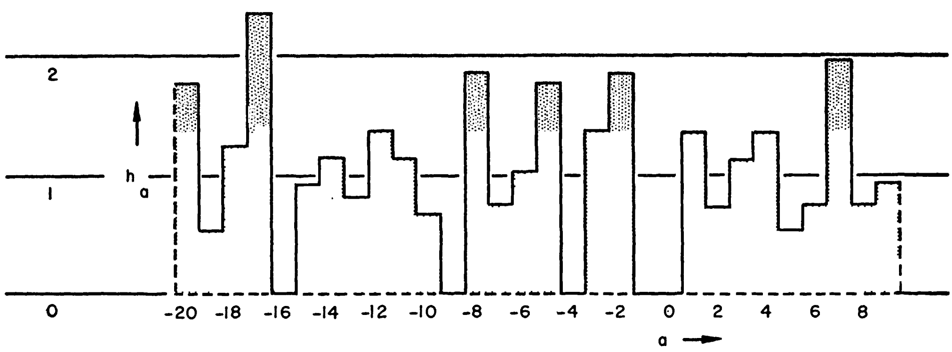

FIG. 1—Distribution of primality in the neighborhood of square numbers.

6. Next, two remarks concerning the computation of h_a. If $a < 0$, both $L_a(1)$ and $L_a(2)$ are available in closed form [2]. Thus, one may calculate h_a with moderate accuracy quite simply by the use of [1, eq. (18)]. The covergence of the remaining factor is fairly rapid, and particularly so if $-a$ is a quadratic residue of only a few small primes.

For example, up to $p = 43$, $(17/p) = +1$ only for 13 and 19. Thus, using the theory in [1] and [2], we determine that

$$h_{-17} = \frac{145\pi^2}{289 \log (4 + \sqrt{17})}\left(1 - \frac{2}{13\cdot 12^2}\right)\left(1 - \frac{2}{19\cdot 18^2}\right) \prod_{\substack{p \geqq 43 \\ \left(\frac{17}{p}\right)=+1}} \left(1 - \frac{2}{p\cdot(p-1)^2}\right).$$

Ignoring the last factor we obtain $h_{-17} \approx 2.3606$ and therefore

$$P_{-17}(N) \sim 1.1803 \int_2^N \frac{dn}{\log n}.$$

(The Hardy-Littlewood conjecture is particularly frustrating in a case such as this, where the sequence $n^2 - 17$ has even *more* primes than the sequence n, since we are unable to prove that $P_{-17}(N) \to \infty$ even in the weakest possible way.)

By similar computations we can compute three decimal place values of h_a for other negative a, and we present some such values in Table 4.

TABLE 4

$h_{-5} = 1.773(3)$	$h_{-14} = 1.151(7)$
$h_{-7} = 0.757(4)$	$h_{-15} = 0.911(8)$
$h_{-10} = 0.671(1)$	$h_{-17} = 2.360(6)$
$h_{-11} = 1.148(0)$	$h_{-19} = 0.544(2)$
$h_{-13} = 0.807(2)$	

7. There also is an interesting *complementary* formula for h_a. Here the convergence is fastest if $-a$ is a quadratic residue for *many* small primes. Since, by (2) we have

$$h_a = \prod_{p \nmid a} \left(1 - \left(\frac{-a}{p}\right)\frac{1}{p-1}\right),$$

we now define

$$h_a^* = \prod_{p \nmid a} \left(1 + \left(\frac{-a}{p}\right)\frac{1}{p-1}\right). \tag{4}$$

Then

$$h_a h_a^* \prod_{p \mid a} \left(1 - \frac{1}{(p-1)^2}\right) = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) = c_2. \tag{5}$$

Since c_2, the so-called "twin-prime constant," is accurately known [3], we may compute h_a^* by (4), and utilize (5) to evaluate h_a. In this way we interchange the roles of the quadratic residues and nonresidues. Using the notation of [1] we therefore obtain

$$h_a = \frac{0.6601618158}{\prod_{p \mid a}\left(1 - \frac{1}{(p-1)^2}\right) L_a(1)} \prod_{s=2}^{\infty} \left(\frac{\zeta_a(s)}{L_a(s)}\right)^{b(s)} . \tag{6}$$

We note that the right side of (6) converges monotonically increasing, while that in [1, eq. (18)] converges monotonically decreasing. Thus h_a may be bounded.

To illustrate (6), and such bounds, we note that $(19/q) = -1$ for $q = 7, 11, 13, 23$, while $(19/p) = +1$ for $p = 3, 5$, and 17. Therefore we find

$$h_{-19} = \frac{0.6601618158}{\log(170 + 39\sqrt{19})} \cdot \frac{29160}{6137} \cdot \frac{252}{250} \cdot \frac{1100}{1098} \cdot \frac{1872}{1870} \cdot \frac{11132}{11130} \cdot \prod_{\substack{q \geqq 29 \\ \left(\frac{19}{q}\right) = -1}} \left(1 - \frac{2}{q(q-1)^2}\right)^{-1} .$$

Ignoring the last factor, we obtain

$$0.54411 < h_{-19} .$$

On the other hand, from [1, eq. (18)], we have

$$h_{-19} = \frac{\pi^2}{\log(170 + 39\sqrt{19})} \cdot \frac{2715}{6859} \cdot \frac{10}{12} \cdot \frac{78}{80} \cdot \frac{4350}{4352} \cdot \prod_{\substack{p \geqq 31 \\ \left(\frac{19}{p}\right) = +1}} \left(1 - \frac{2}{p(p-1)^2}\right),$$

and therefore

$$h_{-19} < 0.54431.$$

8. Finally, these bounds suggest a weakened version of the Hardy-Littlewood conjecture that may be less unattainable. We have, for all a,

$$\frac{c_2}{L_a(1)} < h_a < \frac{\pi^2}{8L_a(1)} , \tag{7}$$

or, with numerical coefficients,

$$\frac{0.6601618158}{L_a(1)} < h_a < \frac{1.233700550}{L_a(1)} . \tag{7a}$$

While these are not very close bounds, they are valid for all a, and they suggest, for sufficiently large N, the inequalities:

$$\frac{c_2}{2L_a(1)} \int_2^N \frac{dn}{\log n} < P_a(N) < \frac{\pi^2}{16L_a(1)} \int_2^N \frac{dn}{\log n} . \tag{8}$$

The best coefficients that have been proven are $2h_a$ on the right and 0 on the left. For the former, see the use, by Bateman and Stemmler [4], of A. Selberg's sieve method. Equation (8) is, of course, reminiscent of the old Chebyshev inequalities for $\pi(N)$, and historical precedent therefore suggests that an investigation of such bounds is in order. While the lower bound would be particularly important, the improved upper bound would also be useful.

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An Approximation to the Fermi Integral $F_{1/2}(x)$

By H. Werner and G. Raymann

The Fermi Integral as defined, for instance, in the *Handbuch der Physik*, Bd. XX, S. 58 [1], is given by

$$F_p(x) = \int_0^\infty \frac{t^p}{e^{t-x} + 1}\, dt. \tag{1}$$

The function $F_{1/2}(x)$ has for negative values of x an expansion of the form

$$F_{1/2}(x) = \frac{\sqrt{\pi}}{2} \sum_{\nu=1}^{\infty} (-1)^{\nu-1} \cdot \frac{e^{\nu x}}{\nu^{3/2}}, \tag{2}$$

and for large positive x the asyimptotic expansion

$$\begin{aligned} F_{1/2}(x) \sim x^{3/2} \Bigg[\frac{2}{3} + \frac{\pi^2}{12 \cdot x^2} + \binom{\frac{1}{2}}{3} \cdot \frac{7}{60} \cdot \frac{\pi^4}{x^4} + \cdots \\ + \binom{\frac{1}{2}}{2n-1} \frac{2^{2n-1} - 1}{n} \mid B_{2n} \mid \cdot \frac{\pi^{2n}}{x^{2n}} + \cdots \Bigg]; \end{aligned} \tag{3}$$

compare [2], formulas (10) and (12);

B_{2n} are the Bernoulli numbers, given for example in [3], page 298. We obtained Chebyshev approximations to $F_{1/2}(x)$, based upon the table by McDougall and Stoner [4]. This table was subtabulated by interpolation with a fifth-degree polynomial. The approximations are

$$\begin{aligned} F^*_{1/2}(x) &= e^x \sum_{\nu=0}^{5} a_\nu e^{\nu x} && \text{for} \quad -\infty < x \leqq +1, \\ F^*_{1/2}(x) &= x^{3/2} \left[\frac{2}{3} + \sum_{\nu=0}^{5} \frac{b_\nu}{x^{2\nu+2}} \right] && \text{for} \quad +1 < x < +\infty, \end{aligned} \tag{4}$$

Received March 27, 1961. This note was prepared at the AEG Forschungs-Institut, Frankfurt (Main).