Tables For The Evaluation Of $\int_0^\infty x^\beta e^{-x} f(x)\, dx$ By Gauss-Laguerre Quadrature

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Abstract. Tables of abscissae and weight coefficients to fifteen places are presented for the Gauss-Laguerre quadrature formula $\int_0^\infty x^\beta e^{-x} f(x)\, dx \sim \sum_{k=1}^{n} H_k f(a_k)$ for $\beta = -\frac{1}{4}, -\frac{1}{2}$, and $-\frac{3}{4}$ and $n = 1(1)15$.

1. Introduction. The n-point Gauss quadrature formula for evaluating the definite integral $\int_0^\infty x^\beta e^{-x} f(x)\, dx$ is (see, for example, [1])

$$\int_0^\infty x^\beta e^{-x} f(x)\, dx = \sum_{k=1}^{n} H_k f(a_k) + E_n,$$

where a_k is the kth zero of the Laguerre polynomial $L_n^\beta(x)$, H_k is the corresponding weight coefficient, and E_n is the truncation error. If the normalization of the Laguerre polynomials is chosen so that

$$L_n^\beta(x) = \sum_{m=0}^{n} \binom{n+\beta}{n-m} \frac{(-x)^m}{m!}$$

then the weight coefficients are given by

$$H_k = \frac{\Gamma(n+\beta+1) a_k}{n![(n+1)L_{n+1}^\beta(a_k)]^2}, \qquad (k = 1, 2, \cdots, n) \tag{1}$$

or the alternative formula

$$H_k = \frac{\Gamma(n+\beta+1)}{n!\, a_k \left[\frac{d}{dx} L_n^\beta(a_k)\right]^2} \qquad (k = 1, 2, \cdots, n) \tag{2}$$

and the truncation error by

$$E_n = \frac{n!\,\Gamma(n+\beta+1)}{(2n)!} f^{(2n)}(\xi).$$

For $\beta = 0$, a 12-place table has been prepared by Salzer and Zucker [2] for values of n ranging from 1 to 15. A short table has also been prepared by Burnett [3] for n, $\beta = 2, 3, 4$. Both of these tables have been reproduced in [1]. More recently, Rabinowitz and Weiss [4] have prepared tables to 18 significant digits for several integral values of β and n.

The tables presented here are designed to fill the need for integrands involving certain fractional powers of β. The roots and weight coefficients are given to 15 places for $\beta = -\frac{1}{4}, -\frac{1}{2}$, and $-\frac{3}{4}$ and for $n = 1(1)15$. It should be noted that for

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$\beta = -\frac{1}{2}$ the Gauss-Laguerre quadrature formula is directly related to the Gauss-Hermite quadrature formula, since the Laguerre polynomial of order $-\frac{1}{2}$ and degree n in x^2 is merely a multiple of the Hermite polynomial of degree $2n$ in x (see, for example, [1]).

TABLE I. ABCISSAE AND WEIGHT COEFFICIENTS FOR THE GAUSS-LAGUERRE QUADRATURE FORMULA

$$\int_0^\infty x^{-1/4} e^{-x} f(x)dx \sim \sum_{k=1}^{N} H_k f(a_k)$$

a_k	H_k	$H_k \cdot \exp(a_k)$
	N=1	
0.75000 00000 00000	1.22541 67024 65178	2.59420 71794 76230
	N=2	
0.42712 43444 67705	1.07587 23293 96545	1.64914 36384 64101
3.07287 56555 32295	0.14954 43730 68633	3.23074 75017 97830
	N=3	
0.29934 69776 68357	0.92931 04968 56913	1.25361 90505 66046
2.03993 89987 79068	0.28658 93410 36511	2.20391 21770 41807
5.91071 40235 52575	0.(2)95168 64571 75359	3.51143 28986 92471
	N=4	
0.23055 42735 67546	0.81598 32950 69312	1.02756 59781 22273
1.54191 41073 68344	0.37322 42412 00384	1.74427 36982 35094
4.23305 69607 04901	0.(1)35751 54927 85345	2.46426 91838 24440
8.99447 46583 59209	0.(3)45761 69169 47827	3.68767 62186 46727
	N=5	
0.18751 15165 66305	0.72848 46471 50010	0.87873 03375 29035
1.24301 39853 18345	0.42348 79614 74704	1.46782 80537 79906
3.34155 40660 74481	0.(1)70352 68628 74916	1.98837 90409 78366
6.75166 77467 61627	0.(2)30727 12884 80510	2.62865 76456 55227
12.22625 26852 79242	0.(4)18694 66816 67895	3.81515 19201 88150
	N=6	
0.15802 68336 99440	0.65938 18290 67955	0.77226 64300 96141
1.04245 20303 14834	0.45086 00686 64817	1.27871 25784 05021
2.77254 78222 38264	0.10607 62609 78039	1.69715 07607 60404
5.48634 93705 04721	0.(2)88885 64891 21336	2.14547 22653 17926
9.48223 30795 38669	0.(3)20929 37342 74977	2.74686 76167 18750
15.55839 08637 04072	0.(6)68512 88785 64549	3.91467 37444 25026
	N=7	
0.13656 13094 49543	0.60350 36343 91610	0.69181 13979 52602
0.89815 86048 14082	0.46409 82356 50759	1.13939 74504 24671
2.37383 76105 70584	0.13918 94937 12046	1.49468 96606 34001
4.64534 53499 66396	0.(1)17749 20341 99025	1.84767 98338 83035
7.86894 97868 86669	0.(3)86398 60993 51623	2.25916 64298 18023
12.36339 39179 28775	0.(4)12125 97533 51522	2.83837 93377 50396
18.96375 34203 83951	0.(7)23216 17328 41405	3.99617 13631 89639
	N=8	
0.12023 31997 51112	0.55736 92781 85813	0.62857 86736 21423
0.78921 46737 12894	0.46857 43009 09540	1.03164 44443 48223
2.07753 62808 55126	0.16824 66863 03240	1.34341 15012 67311
4.03808 38266 34674	0.(1)28898 18348 80682	1.63903 43963 32350
6.76498 92788 69790	0.(2)22580 83788 30631	1.95766 13503 64862
10.42678 75702 41939	0.(4)69546 10642 17201	2.34730 28524 25216
15.35788 12262 35472	0.(6)62294 26466 51578	2.91266 90027 08667
22.42527 39436 98994	0.(9)74114 08084 41323	4.06512 08009 99067

Note:The number in parentheses designates the number of zeros between the decimal point and the first significant figure.

TABLE I. (continued)

a_k	H_k	$H_k \cdot \exp(a_k)$
	N=9	
0.10739 44508 45937	0.51859 92630 06697	0.57739 45930 20612
0.70398 00035 60476	0.46758 47561 68817	0.94535 51079 10793
1.84810 13190 08027	0.19297 40810 65906	1.22495 23250 55226
3.57613 64928 04962	0.(1)41469 99523 08079	1.48193 90113 28977
5.94990 30500 18857	0.(2)45490 64996 71575	1.74554 96428 86997
9.07193 16344 35723	0.(3)23466 33195 74428	2.04331 37570 72879
13.11989 99164 49664	0.(5)48495 08331 22167	2.41878 82298 41196
18.44120 23256 02641	0.(7)29145 75846 95145	2.97500 54883 40743
25.93145 08072 73712	0.(10)22569 50863 49205	4.12484 94049 14763
	N=10	
0.(1)97034 09237 94910	0.48552 55709 45590	0.53499 96261 04034
0.63544 11414 38348	0.46318 26268 49682	0.87442 15309 05008
1.66492 09791 93740	0.21362 56612 96518	1.12906 62201 29222
3.21165 98812 87449	0.(1)54714 98209 05935	1.35803 95736 70919
5.31873 70329 77139	0.(2)77654 92856 35028	1.58513 83333 79665
8.05413 46553 70353	0.(3)58122 22892 52143	1.82897 80692 46425
11.52656 57824 80866	0.(4)20843 02466 10583	2.11292 77990 16324
15.92106 18565 25647	0.(6)30184 68995 18253	2.47865 46996 43792
21.59631 72944 01876	0.(8)12649 70162 32912	3.02859 85636 10315
29.47412 72839 45092	0.(12)66140 61032 38781	4.17752 47969 18064
	N=11	
0.(1)88497 42523 93362	0.45694 85721 92504	0.49923 06814 00587
0.57911 24073 37242	0.45666 88117 59281	0.81490 44236 45476
1.51514 79859 77567	0.23065 51209 41987	1.04950 25804 02527
2.91612 95590 85023	0.(1)68062 20626 85712	1.25708 60264 49263
4.81341 74827 59961	0.(1)11839 56866 64997	1.45806 38162 91429
7.25481 70039 65659	0.(2)11777 85375 09600	1.66645 33062 10863
10.31284 21061 33851	0.(4)62988 20181 98374	1.89701 00415 48549
14.10108 72261 02414	0.(5)16318 80536 46445	2.17126 45099 29840
18.81081 36892 20167	0.(7)17127 25726 79620	2.52999 93327 48339
24.81084 76736 80316	0.(10)51606 79867 03971	3.07553 84103 80859
33.04728 74404 98465	0.(13)18773 04174 17480	4.22463 52952 25459
	N=12	
0.(1)81341 74682 89478	0.43198 47895 82901	0.46859 18460 00142
0.53198 78171 87755	0.44888 08007 87421	0.76413 55482 43572
1.39032 92119 96853	0.24455 97868 92442	0.98219 41702 49929
2.67134 11348 36481	0.(1)81110 95088 04867	1.17281 14759 77219
4.39863 37095 14387	0.(1)16647 23621 41981	1.35408 05187 47342
6.60730 09190 79570	0.(2)20761 34263 80793	1.53733 94271 07768
9.34863 87675 55780	0.(3)15091 77920 04223	1.73301 64005 09255
12.69870 68628 05904	0.(5)59700 04681 81295	1.95412 79185 92914
16.77515 64946 47187	0.(6)11514 55052 48887	2.22128 80921 88296
21.77471 85777 38606	0.(9)89973 00086 47021	2.57485 23424 56110
28.07549 07798 60442	0.(11)19986 73787 16358	3.11725 72171 49009
36.64635 39779 48087	0.(15)51860 69499 56289	4.26724 56763 77220

TABLE I. (continued)

a_k	H_k	$H_k \cdot \exp(a_k)$
	N=13	
0.(1)75256 94975 13055	0.40996 95797 12328	0.44201 32729 26994
0.49197 58561 09853	0.44036 47508 58990	0.72023 62036 49030
1.28466 20374 98612	0.25581 16878 65549	0.92436 18587 67703
2.46506 99450 46952	0.(1)93598 87745 47659	1.10112 57399 82658
4.05145 76689 15234	0.(1)22040 68471 69280	1.26692 46656 69027
6.07045 14338 13352	0.(2)33066 84732 61061	1.43138 46334 50552
8.55992 50625 09374	0.(3)30709 14160 57276	1.60249 20999 63391
11.57389 79336 77712	0.(4)16832 16309 71986	1.78903 95326 94695
15.19134 04165 42934	0.(6)50605 64302 05390	2.00315 36522 71924
19.53340 37331 47340	0.(8)74440 74118 72474	2.26495 79457 67152
24.80174 27980 83576	0.(10)44273 82170 14949	2.61461 07986 99561
31.38306 12909 71120	0.(13)74042 01999 03227	3.15477 65386 95040
40.26775 48739 32637	0.(16)13996 67043 61178	4.30614 32907 18488
	N=14	
0.(1)70019 32753 58374	0.39039 33765 89349	0.41870 81828 18740
0.45757 55271 39819	0.43148 01095 94078	0.68184 13053 35317
1.19402 47226 58879	0.26483 11850 34078	0.87403 22795 02873
2.28877 40165 68711	0.10536 76482 27276	1.03922 41070 04988
3.75625 56780 63075	0.(1)27870 14767 59414	1.19250 54801 73131
5.61720 71364 12356	0.(2)48790 59003 02563	1.34232 64045 64355
7.90052 18527 82609	0.(3)55403 49028 60683	1.49516 86070 73682
10.64624 48263 11831	0.(4)39431 44527 28790	1.65747 91597 71874
13.91061 58751 48442	0.(5)16705 25498 56793	1.83720 16359 28725
17.77508 42578 68801	0.(7)39019 16224 65810	2.04596 85297 14964
22.36390 48361 10144	0.(9)44657 29308 13467	2.30363 07058 04425
27.88324 59619 25794	0.(11)20594 48330 85566	2.65027 38979 61100
34.72788 25760 28264	0.(14)26394 06001 95595	3.18884 91181 79522
43.90864 34054 45437	0.(18)37016 97405 95552	4.34192 63599 63870
	N=15	
0.(1)65463 43791 96718	0.37285 88900 32600	0.39808 41738 78956
0.42768 12695 81156	0.42246 45748 03116	0.64793 27215 82153
1.11540 56095 25728	0.27198 06452 38186	0.82976 00113 90182
2.13629 03095 53862	0.11633 34187 38067	0.98510 74063 68602
3.50195 42818 14086	0.(1)33996 58377 74642	1.12801 45424 84820
5.22889 69990 40331	0.(2)67854 50385 71934	1.26607 60471 56155
7.33973 88489 56464	0.(3)91201 09397 47528	1.40477 93951 53081
9.86515 04211 88924	0.(4)80484 86265 83588	1.54915 40201 12816
12.84694 16773 16643	0.(5)44908 43881 66099	1.70484 08279 97837
16.34322 69876 66574	0.(6)15004 56970 03404	1.87930 31535 24385
20.43762 68844 92296	0.(8)27728 40560 75822	2.08388 36985 67799
25.25720 54392 18028	0.(10)25109 13570 48455	2.33828 10736 25637
31.01232 33502 32081	0.(13)91216 12780 79384	2.68257 90846 02620
38.10538 41690 71322	0.(16)90971 18018 91036	3.22004 47285 57758
47.56671 03144 22833	0.(20)96166 58309 90149	4.37505 98374 42590

TABLE II. ABCISSAE AND WEIGHT COEFFICIENTS FOR THE GAUSS-LAGUERRE QUADRATURE FORMULA

$$\int_0^\infty x^{-1/2} e^{-x} f(x)\,dx \simeq \sum_{k=1}^{N} H_k f(a_k)$$

a_k	H_k	$H_k \cdot \exp(a_k)$
	N=1	
0.50000 00000 00000	1.77245 38509 05516	2.92228 23653 22278
	N=2	
0.27525 51286 08411	1.60982 81800 11026	2.11992 89657 89938
2.72474 48713 91589	0.16262 56708 94490	2.48045 16353 91632
	N=3	
0.19016 35091 93488	1.44925 91904 48785	1.75280 26688 72461
1.78449 27485 43252	0.31413 46406 45713	1.87116 11152 62361
5.52534 37422 63260	0.(2)90600 19811 01769	2.27381 66653 49049
	N=4	
0.14530 35215 03317	1.32229 40251 16483	1.52908 82573 03458
1.33909 72881 26361	0.41560 46516 29784	1.58578 00967 72803
3.92696 35013 58287	0.(1)34155 96601 48270	1.73350 52131 26762
8.58863 56890 12034	0.(3)39920 81444 22735	2.14386 02884 95960
	N=5	
0.11758 13202 11778	1.22172 52674 70652	1.37416 37079 02547
1.07456 20124 36904	0.48027 72221 64629	1.40659 26462 09812
3.08593 74437 17550	0.(1)67748 78891 09621	1.48288 38638 87130
6.41472 97336 62030	0.(2)26872 91493 56247	1.64133 22528 09633
11.80718 94899 71737	0.(4)15280 86571 04652	2.05090 33827 31474
	N=6	
0.(1)98747 01406 84812	1.14027 04725 24959	1.25861 57487 38986
0.89830 28345 69618	0.52098 46205 28322	1.27924 24640 40513
2.55258 98026 68171	0.10321 59712 31768	1.32532 55465 33743
5.19615 25300 54466	0.(2)78107 81169 25812	1.41044 07322 24440
9.12424 80375 31179	0.(3)17147 37408 71757	1.57328 78789 26645
15.12995 97811 08085	0.(6)53171 03368 71260	1.97939 80941 84596
	N=7	
0.(1)85115 44299 75940	1.07281 18194 24180	1.16812 33810 43992
0.77213 79200 42777	0.54621 12181 28493	1.18221 33340 86325
2.18059 18884 50459	0.13701 10684 46930	1.21275 94782 52195
4.38979 28867 31014	0.(1)15700 10945 29159	1.26580 12129 44666
7.55409 13261 01784	0.(3)71018 52271 03847	1.35541 35183 84794
11.98999 30398 23879	0.(5)94329 68710 03783	1.51997 41747 95132
18.52827 74958 52492	0.(7)17257 18233 62503	1.92175 74060 51319
	N=8	
0.(1)74791 88259 68183	1.01585 89580 33227	1.09475 04100 75688
0.67724 90876 49289	0.56129 49170 57067	1.10488 39147 34919
1.90511 36350 31428	0.16762 00827 97972	1.12643 56581 76400
3.80947 63614 84907	0.(1)25760 62307 10199	1.16249 45508 01728
6.48314 54286 27170	0.(2)18645 68017 24836	1.21947 39165 11995
10.09332 36752 21343	0.(4)54237 20185 07576	1.31151 13457 52235
14.97262 70884 26393	0.(6)46419 61689 73042	1.47649 12445 55363
21.98427 28409 62651	0.(9)53096 14948 02236	1.87374 89857 68139

TABLE II. (continued)

a_k	H_k	$H_k \cdot \exp(a_k)$
	N=9	
0.(1)66702 23095 81944	0.96699 13894 50911	1.03369 16729 63243
0.60323 63570 81749	0.56961 45713 39959	1.04126 98933 52129
1.69239 50797 93179	0.19460 34952 82631	1.05717 88858 37602
3.36917 62702 43269	0.(1)37280 08477 50893	1.08315 73573 24265
5.69442 33429 57755	0.(2)37770 45260 53684	1.12255 80910 99609
8.76975 67302 68602	0.(3)18362 25373 58588	1.18190 60069 26216
12.77182 53548 69194	0.(5)36213 08962 18686	1.27526 03440 12320
18.04650 54677 28980	0.(7)20934 41159 15842	1.43998 67662 10628
25.48597 91660 99077	0.(10)15656 39954 42318	1.83278 70751 03831
	N=10	
0.(1)60192 06314 95879	0.92448 73392 01220	0.98184 30013 33492
0.54386 75002 94646	0.57335 10107 25668	0.98768 67705 44106
1.52294 41054 04444	0.21803 44120 40047	0.99984 17426 72581
3.02251 33764 51574	0.(1)49621 04177 49272	1.01935 80542 34916
5.08490 77500 98524	0.(2)64875 46684 47572	1.04816 07018 97115
7.77743 92315 25445	0.(3)45667 72720 32708	1.08970 34847 29040
11.20813 02043 48663	0.(4)15605 11295 70641	1.15052 48857 05006
15.56116 33321 89350	0.(6)21721 38741 53856	1.24455 73923 82825
21.19389 20963 01541	0.(9)87986 81984 54636	1.40866 59223 53885
29.02495 03402 36226	0.(12)44587 87291 06830	1.79718 39229 06383
	N=11	
0.(1)54839 86957 88185	0.88709 04528 69919	0.93709 70225 91985
0.49517 41233 50356	0.57394 28664 93814	0.94171 62223 11077
1.38465 57400 84600	0.23820 47219 17565	0.95125 88570 96625
2.74191 99401 06702	0.(1)62280 74176 88477	0.96639 45592 28331
4.59773 77004 85711	0.(2)99567 98670 10329	0.98830 69244 19434
6.99939 74695 28836	0.(3)92977 01017 68504	1.01900 25582 15213
10.01890 82759 57234	0.(4)47310 25710 50209	1.06196 91167 20897
13.76930 58661 01691	0.(5)11768 57512 66020	1.12371 82685 81661
18.44111 96809 78192	0.(7)11933 98197 21193	1.21804 09659 39852
24.40196 12423 87043	0.(10)34886 78015 09599	1.38133 17798 61594
32.59498 00914 40815	0.(13)12334 36684 88081	1.76578 10531 77138
	N=12	
0.(1)50361 88911 72940	0.85386 23277 37398	0.89796 56906 52452
0.45450 66815 63780	0.57235 90706 92886	0.90169 22016 39993
1.26958 99401 03961	0.25547 92435 69118	0.90935 09542 70243
2.50984 80972 32128	0.(1)74890 94100 64615	0.92138 78192 59494
4.19841 56448 78413	0.(1)14096 71162 01453	0.93856 97712 52978
6.36997 53880 30635	0.(2)16473 84965 37683	0.96214 44056 66107
9.07543 42309 61203	0.(3)11377 38327 28088	0.99415 33857 31504
12.39044 79638 09471	0.(5)43164 91409 80467	1.03808 77351 66066
16.43219 50876 75313	0.(7)80379 42349 88286	1.10041 50947 95685
21.39675 59361 66109	0.(9)60925 08539 97513	1.19478 34008 80088
27.66110 87798 46090	0.(11)13169 24048 61563	1.35714 93245 80215
36.19136 03606 15602	0.(15)33287 36992 97822	1.73775 11301 83380

TABLE II. (continued)

a_k	H_k	$H_k \cdot \exp(a_k)$
	N=13	
0.(1)46560 08324 50248	0.82408 73011 80739	0.86336 41458 62896
0.42002 74064 01214	0.56926 44823 53569	0.86642 24023 15866
1.17231 07732 77780	0.27022 66558 23576	0.87268 25391 63063
2.31454 08643 49434	0.(1)87196 45443 45016	0.88245 21216 80923
3.86458 50382 28159	0.(1)18795 80258 23192	0.89624 93048 01149
5.84873 48113 06343	0.(2)26381 29444 64771	0.91489 10010 41484
8.30455 34899 85900	0.(3)23245 94032 06219	0.93965 59624 89779
11.28575 09935 17638	0.(4)12206 58343 47920	0.97259 82965 52898
14.87096 03775 25401	0.(6)35402 12674 79471	1.01719 77366 99941
19.18091 94856 10456	0.(8)50489 88068 98107	1.07987 08919 42139
24.41669 23330 56517	0.(10)29219 99867 96322	1.17412 66307 79573
30.96393 82747 46795	0.(13)47662 97318 74432	1.33551 46629 08603
39.81042 60687 49338	0.(17)87938 32189 50780	1.71248 40599 42642
	N=14	
0.(1)43292 03573 97735	0.79720 94356 52903	0.83248 02184 91461
0.39042 09260 42032	0.56512 27825 18777	0.83502 69065 72399
1.08896 58675 69270	0.28278 92195 73910	0.84022 32921 88656
2.14779 94705 82232	0.(1)99029 77857 97963	0.84828 78884 06087
3.58102 82499 91771	0.(1)23936 84642 87096	0.85958 28499 06605
5.40911 23306 16460	0.(2)39146 62588 81798	0.87466 54515 27500
7.66069 11156 10085	0.(3)42123 62000 48065	0.89437 89423 39797
10.37556 30097 70052	0.(4)28691 00845 94288	0.92001 64132 47248
13.60971 14293 90236	0.(5)11715 43944 19860	0.95363 27504 40262
17.44429 44757 04189	0.(7)26513 65003 08342	0.99868 91944 94002
22.00319 67669 14923	0.(9)29517 06336 55538	1.06154 91030 64781
27.49204 15048 43852	0.(11)13278 87342 98193	1.15558 83454 50543
34.30462 05093 73083	0.(14)16631 87590 24137	1.31597 79936 61231
43.44926 23078 52043	0.(18)22802 78695 80735	1.68951 78822 56733
	N=15	
0.(1)40452 70430 45753	0.77278 97790 83628	0.80469 21334 03806
0.36472 06450 51408	0.56026 18616 78425	0.80683 96338 49608
1.01674 60688 57496	0.29347 16950 81780	0.81121 02466 51369
2.00371 89531 33923	0.11028 83537 40469	0.81796 31500 07063
3.33698 32057 34510	0.(1)29407 65940 96534	0.82735 87272 22278
5.03280 52776 25116	0.(2)54758 44946 13532	0.83979 00074 73648
7.11359 37697 29875	0.(3)69662 02486 37371	0.85583 61258 65487
9.60981 72843 04444	0.(4)58774 50457 84598	0.87635 40453 05367
12.56308 23699 48498	0.(5)31581 89774 64942	0.90264 20719 82377
16.03128 41080 73975	0.(6)10217 04490 15519	0.93674 96251 29458
20.09778 53347 55927	0.(8)18357 16084 87571	0.98211 59916 65765
24.88931 24751 56550	0.(10)16212 37259 49261	1.04505 13786 62709
30.61571 74008 99492	0.(13)57572 14161 09741	1.13880 53838 99281
37.67847 17842 05299	0.(16)56206 67205 50181	1.29819 59631 08533
47.10550 86182 18914	0.(20)58165 09400 26245	1.66849 49420 25524

TABLE III. ABCISSAE AND WEIGHT COEFFICIENTS FOR THE GAUSS-LAGUERRE QUADRATURE FORMULA

$$\int_0^\infty x^{-3/4} e^{-x} f(x) dx \sim \sum_{k=1}^{N} H_k f(a_k)$$

a_k	H_k	$H_k \cdot \exp(a_k)$
	N=1	
0.25000 00000 00000	3.62560 99082 21908	4.65537 52731 51840
	N=2	
0.13196 60112 50105	3.43422 69970 47146	3.91869 18028 83499
2.36803 39887 49895	0.19138 29111 74762	2.04327 70202 76164
	N=3	
0.(1)89682 15690 63772	3.24393 87634 47653	3.54830 63889 47662
1.52744 70442 32944	0.37265 78935 90914	1.71661 20134 44811
5.13287 07988 60679	0.(2)90132 51183 34119	1.52777 33797 59560
	N=4	
0.(1)67925 85575 57447	3.08969 94954 25830	3.30686 19596 57582
1.13717 85166 85784	0.50145 73316 17305	1.56352 32369 86930
3.61792 70673 49276	0.(1)34094 06814 87093	1.27035 34933 24859
8.17696 85602 09195	0.(3)35901 30300 64269	1.27738 71281 81308
	N=5	
0.(1)54666 24082 21539	2.96406 33525 36479	3.13060 82727 95961
0.90803 85112 16728	0.59092 56178 90062	1.46517 30870 77509
2.82938 55747 82284	0.(1)68185 62440 39150	1.15472 71156 63980
6.07468 94151 36615	0.(2)24225 29604 49910	1.05310 86629 39056
11.38322 02580 42220	0.(4)12783 78695 27085	1.12286 94566 54552
	N=6	
0.(1)45738 46887 45652	2.85950 04515 80845	2.99332 68004 64688
0.75652 09910 18956	0.65387 47519 62333	1.39330 90568 34898
2.33277 47212 40585	0.10501 50577 52041	1.08233 76462 71251
4.90438 29130 17239	0.(2)70754 47946 51461	0.95433 39556 61257
8.76328 12419 57814	0.(3)14377 85241 33990	0.91947 24224 77684
14.69730 16638 90840	0.(6)42045 60410 57221	1.01549 45763 88438
	N=7	
0.(1)39317 63548 38735	2.77067 03312 62886	2.88177 64394 66983
0.64865 48953 46562	0.69892 31918 25546	1.33701 62733 96839
1.98810 35172 01898	0.14109 25065 29910	1.03021 13631 74735
4.13359 89124 27036	0.(1)14318 94278 21993	0.89353 20848 88644
7.23743 52581 17768	0.(3)59744 60275 16763	0.83076 24788 81412
11.61379 87096 69502	0.(5)74767 62941 45516	0.82703 19824 81570
18.08909 10717 53362	0.(7)13030 90851 92778	0.93533 48819 61205
	N=8	
0.(1)34477 73946 40940	2.69388 23811 28369	2.78838 10453 65402
0.56786 06512 10971	0.73163 28384 64761	1.29095 74773 44852
1.73380 97161 62386	0.17479 22836 49657	0.98970 61041 18400
3.58082 29874 06068	0.(1)23683 09161 98073	0.85029 58495 06236
6.20028 33693 18819	0.(2)15758 63871 00059	0.77672 54803 17444
9.75800 54150 99088	0.(4)43097 71850 89928	0.74525 03461 24900
14.58477 40368 89801	0.(6)35138 42384 63959	0.75834 90745 32394
21.53996 60844 48773	0.(9)38556 56065 39643	0.87254 27164 86626

TABLE III. (continued)

a_k	H_k	$H_k \cdot \exp(a_k)$
	N=9	
0.(1)30698 86435 38197	2.62653 93174 58659	2.70842 15062 70404
0.50504 32964 74127	0.75563 32293 04299	1.25212 75312 69966
1.53802 20591 17957	0.20550 49875 62283	0.95670 23921 85110
3.16256 36390 73761	0.(1)34572 91461 07966	0.81699 60007 95720
5.43845 97639 22421	0.(2)32102 91256 10744	0.73864 79577 61518
8.46637 97776 07115	0.(3)14640 57805 06960	0.69576 12596 24773
12.42191 36357 44743	0.(5)27469 96771 97004	0.68175 24318 32725
17.64939 17104 45324	0.(7)15241 49374 21216	0.70479 19920 29925
25.03752 72532 60732	0.(10)10990 41120 78605	0.82162 54223 10382
	N=10	
0.(1)27666 55867 07972	2.56676 55577 90772	2.63877 06006 67354
0.45478 44226 05949	0.77334 79703 44341	1.21866 77411 44279
1.38242 57611 58599	0.23313 28349 73219	0.92893 07183 60218
2.83398 00120 92697	0.(1)46436 74708 95670	0.79003 01592 71016
4.85097 14487 64914	0.(2)55491 23502 03625	0.70953 61427 72681
7.50001 09426 42825	0.(3)36564 66626 77638	0.66111 19090 97386
10.88840 80238 34404	0.(4)11868 79857 10245	0.63559 76470 80018
15.19947 80442 37603	0.(6)15844 10942 05678	0.63229 14008 15646
20.78921 46210 70107	0.(9)61932 66726 79684	0.66154 74158 11683
28.57306 01649 22106	0.(12)30377 59926 51750	0.77924 82029 39204
	N=11	
0.(1)25179 46133 08897	2.51317 19430 87850	2.57725 56699 58764
0.41365 01260 33511	0.78643 57877 87516	1.18934 87914 08449
1.25569 13731 97123	0.25781 87298 84105	0.90501 19190 49031
2.56850 48683 60841	0.(1)58823 92300 57547	0.76743 47767 07949
4.38222 21869 10301	0.(2)85745 36376 85480	0.68609 70648 46262
6.74359 09253 55378	0.(3)74801 95764 97544	0.63477 13122 21448
9.72409 81589 96743	0.(4)36099 77825 52870	0.60342 97339 29383
13.43620 72707 43373	0.(6)86028 63687 14274	0.58872 66789 82420
18.06970 34327 27263	0.(8)84148 62506 45160	0.59240 60903 63025
23.99096 96631 30710	0.(10)23835 16025 78215	0.62569 66308 95124
32.14018 25332 13868	0.(14)81815 15671 71140	0.74325 47410 56718
	N=12	
0.(1)23102 65376 35139	2.46470 56914 01417	2.52230 97751 44489
0.37935 66724 22353	0.79605 87051 08553	1.16331 57428 83702
1.15041 19328 48111	0.27980 63187 19897	0.88404 64246 33102
2.34927 40135 88590	0.(1)71390 73605 83163	0.74802 92843 68836
3.99856 00495 71689	0.(1)12225 76741 13150	0.66654 38020 03283
6.13252 25283 38813	0.(2)13323 27324 01608	0.61366 53880 65288
8.80166 58898 42439	0.(4)87141 39386 07227	0.57908 11522 44871
12.08123 15274 63674	0.(5)31636 04948 31739	0.55846 30362 16863
16.08791 60473 21063	0.(7)56781 75865 57760	0.55093 70815 55076
21.01714 03815 80805	0.(9)41694 98761 00274	0.55938 64499 53402
27.24475 27410 98319	0.(12)87605 37829 77861	0.59535 57406 41093
35.73406 55621 60630	0.(15)21551 77086 21477	0.71218 11106 77831

TABLE III. (continued)

a_k	H_k	$H_k \exp(a_k)$
	N=13	
0.(1)21342 34275 84664	2.42055 25676 88603	2.47276 80496 20356
0.35032 52295 43789	0.80304 82073 88569	1.13995 03363 06763
1.06153 06703 46692	0.29936 95808 67135	0.86541 53255 64535
2.16501 56086 41061	0.(1)83886 78142 33404	0.73105 13168 18872
3.67822 11595 44482	0.(1)16419 26539 63376	0.64980 77333 06520
5.62708 86089 59671	0.(2)21454 89695 31452	0.59612 86094 58132
8.04886 00306 83873	0.(3)17878 81714 21670	0.55964 71281 95697
10.99692 74644 77245	0.(5)89734 80851 08276	0.53563 11874 08216
14.54957 95697 34920	0.(6)25062 98260 65882	0.52219 76028 48635
18.82713 43854 34178	0.(8)34610 09164 64354	0.51966 48268 96899
24.03005 84247 71594	0.(10)19470 01505 87874	0.53148 13887 17222
30.54295 81792 27926	0.(13)30950 21290 10534	0.56924 84267 84931
39.35095 83258 76102	0.(17)55690 70926 31006	0.68499 40632 29722
	N=14	
0.(1)19831 30003 51657	2.38007 14721 58048	2.42774 25106 30596
0.32542 89264 55660	0.80801 06466 61854	1.11879 12735 32586
0.98547 25901 57927	0.31677 86929 18489	0.84867 47291 95597
2.00788 56746 41348	0.(1)96136 40563 50146	0.71598 10854 40175
3.40641 81936 78566	0.(1)21063 17077 54687	0.63520 27556 67233
5.20124 01688 30572	0.(2)32020 73881 69780	0.58116 74101 54161
7.42072 82152 96688	0.(3)32543 30123 38128	0.54355 42825 74032
10.10442 73103 08717	0.(4)21163 29905 46133	0.51746 42994 12515
13.30805 87811 01103	0.(6)83147 37759 67036	0.50057 02672 71015
17.11248 45724 22704	0.(7)18208 32941 19256	0.49218 42625 92558
21.64121 43114 12187	0.(9)19697 33497 60737	0.49325 03791 84835
27.09932 09059 51765	0.(12)86368 19556 75461	0.50750 38942 07720
33.87960 60056 62917	0.(14)10564 40502 50458	0.54647 58577 72912
42.98788 30440 44682	0.(18)14150 18667 34415	0.66094 08033 36074
	N=15	
0.(1)18520 07901 00734	2.34274 96154 18758	2.38654 17889 59345
0.30384 16926 17680	0.81139 57220 93596	1.09948 54439 56500
0.91963 63045 60125	0.33228 46565 18090	0.83349 55016 79279
1.87224 52173 43486	0.10802 11707 58834	0.70244 87542 09056
3.17271 03197 15959	0.(1)26066 57355 43596	0.62226 37924 86913
4.83705 43317 02695	0.(2)45056 12328 44013	0.56814 58338 42183
6.88746 68596 04685	0.(3)54072 48503 18760	0.52986 49355 50876
9.35420 88904 05215	0.(4)43513 39705 04700	0.50246 32386 15927
12.27867 88974 79813	0.(5)22477 50554 54717	0.48340 44051 58773
15.71854 86393 90621	0.(7)70314 31690 10255	0.47154 45172 59419
19.75691 95767 41943	0.(8)12269 93807 80246	0.46683 51221 70151
24.52017 68467 91694	0.(10)10559 52348 15706	0.47056 73819 43569
30.21765 93557 56172	0.(13)36629 02248 51901	0.48661 78584 17214
37.24990 09353 68249	0.(16)34984 59074 22858	0.52638 46611 82723
46.64243 20535 11588	0.(20)35421 03787 93840	0.63945 81778 13751

In fact the integral

$$I = \int_0^\infty x^{-1/2} e^{-x} f(x)\,dx \tag{3}$$

can be transformed by the substitution $u = x^{1/2}$ into the integral

$$I = \int_{-\infty}^\infty e^{-u^2} f(u^2)\,du, \tag{4}$$

which is of a form suitable for Gauss-Hermite quadrature. However, the tables presented here for $\beta = -\frac{1}{2}$ allow more accuracy to be obtained in evaluating integrals of the form (3) than could be obtained by transforming them to type (4) and utilizing the existing tables for Gauss-Hermite quadrature [5] (duplicated in [1]).

2. Method of Calculation. The Laguerre polynomials and their derivatives were calculated by using the recursion relations [6]

$$L_0^\beta(x) = 1, \quad L_1^\beta(x) = 1 + \beta - x,$$

$$L_n^\beta(x) = \frac{2n + \beta - x - 1}{n} L_{n-1}^\beta(x) - \frac{n + \beta - 1}{n} L_{n-2}^\beta(x) \qquad (n = 2, 3, \cdots)$$

and

$$\frac{d}{dx} L_n^\beta(x) = \frac{1}{x}\,[nL_n^\beta(x) - (n + \beta)L_{n-1}^\beta(x)] \qquad (n = 1, 2, \cdots).$$

The roots and weight coefficients were calculated on the IBM-7090 computer by using a combination of Muller's method, as programmed in the SHARE code B4 RW GRT, and Newton's method. The high degree of accuracy was obtained through the use of revised versions of the SHARE double-precision arithmetic package and input-output routines A1 NR NPRE, A1 NR DICV, and A1 NR DOCV.

The sums and products of the roots were checked by means of the formulas

$$\sum_{k=1}^n a_k = n(n + \beta),$$

and

$$\prod_{k=1}^n a_k = \binom{n+\beta}{n} n!,$$

and the weight coefficients checked by comparison of the calculations from (1) and (2). In all cases, the calculations proved accurate to at least one more significant digit than the rounded values given in the tables. The required gamma functions were calculated with the aid of Gauss' original table reproduced in [7].*

**Added in proof:* A recently published book [8], which is reviewed in *Math. Comp.* v. 17, January, 1963, p. 93, contains tables to eight significant digits for $\beta = -0.90(0.02)0.00$, $\beta = 0.55(0.05)3.00$, $\beta = -\frac{3}{4}, -\frac{1}{4}$, and $\beta = n + \frac{k}{3}$, where $n = -1(1)2$, $k = 1, 2$. The values given here are in agreement with the e'ght-digit ones given there.

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