Evaluation of the Remainder Term in Approximation Formulas by Bernstein Polynomials

By D. D. Stancu

1. Introduction. In this paper [7] we deal with the evaluation of the remainder term in approximation formulas, for functions of one and two variables, by means of Bernstein polynomials.

First, we give a representation of the remainder by definite integrals. Thus, we find the representation (3.4) for the remainder of formula (2.2) and the representation (7.3) for the one corresponding to the formula (6.1). The consequences of these representations are formulas (5.3), (5.4), and (8.2), (8.3), respectively.

Then, we extend Aramă's [1] formula (9.1) to two variables and obtain the representation (9.2). In order to establish this formula, we resorted to a method employed by us, some years ago, in the evaluation of the remainder term of certain interpolation formulas [5]. In the last part we give some numerical approximation formulas which result from formulas (2.2) and (6.1).

2. The Remainder R_m $(f; x)$ of Bernstein's Approximation Formula. Let $f(x)$ be a real function defined on the interval [0, 1] with its second derivative R-integrable on this interval.

The problem is to find an integral expression for the remainder of the approximation formula on the interval [0, 1] of the function $f(x)$ by means of the corresponding Bernstein polynomial

$$B_m(f; x) = \sum_{i=0}^{m} p_{m,i}(x) f\left(\frac{i}{m}\right), \tag{2.1}$$

where

$$p_{m,i}(x) = \binom{m}{i} x^i (1 - x)^{m-i}.$$

The expression

$$R_m(f; x) = f(x) - B_m(f; x)$$

is called the remainder term of the approximation formula

$$f(x) = B_m(f; x) + R_m(f; x), \tag{2.2}$$

by the Bernstein polynomial (2.1).

Since

$$B_m(1; x) = \sum_{i=0}^{m} p_{m,i}(x) = 1, \qquad B_m(x; x) = \frac{1}{m} \sum_{i=0}^{m} i p_{m,i}(x) = x,$$

Received January 26, 1962. Presented to the American Mathematical Society, February 22, 1962.

we have

$$R_m(1; x) = R_m(x; x) = 0,$$

that is, the remainder term $R_m(f; x)$ vanishes for any polynomial of the first degree.

3. The Integral Representation of R_m $(f; x)$. In order to obtain the desired result we shall use the auxiliary function

$$g(f; x) = \int_0^x (x - t)f''(t)\, dt.$$

Since the second derivative of this function is equal to $f''(x)$, the function $h(x) = f(x) - g(f; x)$ represents a polynomial of the first degree. Taking this into account, as well as the linearity of the remainder, we may write

$$\begin{aligned} R_m(h; x) = h(x) - B_m(h; x) &= f(x) - g(f; x) - B_m(f - g; x) \\ &= f(x) - g(f; x) - B_m(f; x) + B_m(g; x) = 0 \end{aligned}$$

hence

$$R_m(f; x) = R_m(g; x). \tag{3.1}$$

Next, we turn to finding a convenient evaluation of $R_m(g; x)$. We have

$$R_m(g; x) = \int_0^x (x - t)f''(t)\, dt - \sum_{i=0}^m p_{m,i}(x) \int_0^{\frac{i}{m}} \left(\frac{i}{m} - t\right) f''(t)\, dt.$$

After certain calculations we obtain

$$\begin{aligned} \sum_{i=1}^m p_{m,i}(x) \int_0^{\frac{i}{m}} \left(\frac{i}{m} - t\right) f''(t)\, dt &= \sum_{i=1}^m p_{m,i}(x) \sum_{k=1}^i \int_{\frac{k-1}{m}}^{\frac{k}{m}} \left(\frac{i}{m} - t\right) f''(t)\, dt \\ &= \sum_{j=1}^m \int_{\frac{j-1}{m}}^{\frac{j}{m}} \left[\sum_{s=j}^m p_{m,s}(x) \left(\frac{s}{m} - t\right) f''(t)\right] dt. \end{aligned}$$

Therefore,

$$\begin{aligned} R_m(g; x) &= \int_0^x (x - t)f''(t)\, dt \\ &\quad - \sum_{j=1}^m \int_{\frac{j-1}{m}}^{\frac{j}{m}} \sum_{s=j}^m p_{m,s}(x) \left(\frac{s}{m} - t\right) f''(t)\, dt = \int_0^1 \varphi_m(x; t) f''(t)\, dt, \end{aligned} \tag{3.2}$$

where, supposing that

$$\frac{k-1}{m} \leqq x \leqq \frac{k}{m} \qquad (k = 1, 2, \cdots, m),$$

we have

$$(3.3)\quad \varphi_m(x;t) = \begin{cases} x - t - \sum_{s=j}^{m}\left(\frac{s}{m} - t\right) p_{m,s}(x), & \\ \qquad\qquad \text{if } t \in \left[\frac{j-1}{m}, \frac{j}{m}\right] \text{ and } 1 \leqq j \leqq k-1 \\ x - t - \sum_{s=k}^{m}\left(\frac{s}{m} - t\right) p_{m,s}(x), & \text{if } t \in \left[\frac{k-1}{m}, x\right] \\ -\sum_{s=k}^{m}\left(\frac{s}{m} - t\right) p_{m,s}(x), & \text{if } t \in \left[x, \frac{k}{m}\right] \\ -\sum_{s=j}^{m}\left(\frac{s}{m} - t\right) p_{m,s}(x), & \\ \qquad\qquad \text{if } t \in \left[\frac{j-1}{m}, \frac{j}{m}\right] \text{ and } k+1 \leqq j \leqq m. \end{cases}$$

It follows from (3.1) and (3.2) that the remainder of Bernstein's formula (2.2) may be represented by the integral as follows

$$(3.4)\qquad R_m(f;x) = \int_0^1 \varphi_m(x;t) f''(t)\, dt,$$

where $\varphi_m(x;t)$ is the kernel, defined for $0 \leqq x, t \leqq 1$ by (3.3).

Thus, formula (2.2) becomes

$$(3.5)\qquad f(x) = B_m(f;x) + \int_0^1 \varphi_m(x;t) f''(t)\, dt.$$

4. Properties of the Kernel $\varphi_m(x,t)$. Considering the identities

$$\sum_{s=0}^{j-1} p_{m,s}(x) + \sum_{s=j}^{m} p_{m,s}(x) = 1,$$

$$\sum_{s=0}^{j-1} s p_{m,s}(x) + \sum_{s=j}^{m} s p_{m,s}(x) = mx,$$

we deduce that

$$x - t - \sum_{s=j}^{m}\left(\frac{s}{m} - t\right) p_{m,s}(x) = x - t - \frac{1}{m}\left(mx - \sum_{s=0}^{j-1} s p_{m,s}(x)\right) + t\left(1 - \sum_{s=0}^{j-1} p_{m,s}(x)\right) = -\sum_{s=0}^{j-1}\left(t - \frac{s}{m}\right) p_{m,s}(x).$$

Thus, we have

$$(4.1)\qquad \varphi_m(x;t) = -\sum_{s=0}^{j-1}\left(t - \frac{s}{m}\right) p_{m,s}(x),$$

for $t \in \left[\frac{j-1}{m}, \frac{j}{m}\right]$ and $1 \leqq j \leqq k-1$, while

$$(4.2)\qquad \varphi_m(x;t) = -\sum_{s=0}^{k-1}\left(t - \frac{s}{m}\right) p_{m,s}(x),$$

for $t \in \left[\frac{k-1}{m}, x\right]$.

It follows from (3.3), (4.1) and (4.2) that $\varphi_m(x; t) \leqq 0$ on the square $D: 0 \leqq x, t \leqq 1$. Hence, $y = \varphi(t) = \varphi_m(x; t)$ represents, for fixed x, a polygonal continuous line which joins points (0, 0) and (0, 1) and *is situated beneath the x-axis.*

5. Other Evaluations of $R_m(f; x)$. Since $\varphi_m(x; t)$ does not change sign on the integration interval [0, 1], the first law of the mean may be applied to (3.5) and we obtain

$$R_m(f; x) = \mu_2(f) \int_0^1 \varphi_m(x; t)\, dt, \tag{5.1}$$

where

$$\inf_{[0,1]} f''(t) \leqq \mu_2(f) \leqq \sup_{[0,1]} f''(t).$$

Because $\varphi_m(x; t)$ does not depend on $f(x)$, let us replace the function $f(x)$ in the formula

$$f(x) = B_m(f; x) + \mu_2(f) \int_0^1 \varphi_m(x; t)\, dt$$

by x^2; we then obtain

$$x^2 = B_m(x^2; x) + 2 \int_0^1 \varphi_m(x; t)\, dt.$$

Since

$$B_m(x^2; x) = \frac{1}{m} \sum_{i=0}^{m} i^2 p_{m,i}(x) = x^2 + \frac{x(1 - x)}{m},$$

it follows that

$$\int_0^1 \varphi_m(x; t)\, dt = -\frac{x(1 - x)}{2m}. \tag{5.2}$$

Thus, Bernstein's approximation formula (3.5) becomes

$$f(x) = B_m(f; x) - \frac{x(1 - x)}{2m} \mu_2(f). \tag{5.3}$$

In the particular case when $f''(x)$ is continuous on [0, 1] this may also be written as

$$f(x) = B_m(f; x) - \frac{x(1 - x)}{2m} f''(\xi), \tag{5.4}$$

where $0 < \xi < 1$.

Remarks. 1. Using the hypothesis that the second derivative $f''(x)$ is continuous on [0, 1], E. Voronowskaja [6] has established the asymptotic formula

$$f(x) = B_m(f; x) - \frac{x(1 - x)}{2m} f''(x) + \frac{\epsilon_m}{m}, ^* \tag{5.5}$$

* In fact, this formula holds if $f(x)$ is bounded on [0, 1] and if $f''(x)$ exists at a point x of [0, 1] (see [4] or [3]).

where $\epsilon_m \to 0$, as $m \to \infty$. There is a close analogy between this formula and formula (5.4).

2. If we take into account the representation of the Bernstein polynomials given by M. Kac [2]

$$B_m(f; x) = \int_0^1 f[\alpha_m(x; t)]\, dt,$$

where

$$\alpha_m(x; t) = \frac{1}{m} \sum_{i=1}^{m} \tilde{\varphi}_i(x; t),$$

and

$$\tilde{\varphi}_1(x; t) = \begin{cases} 1 & \text{if} \quad 0 \leqq t \leqq x \\ 0 & \text{if} \quad x < t \leqq 1, \end{cases}$$

$$\tilde{\varphi}_2(x; t) = \begin{cases} 1 & \text{if} \quad 0 \leqq t \leqq x^2 \quad \text{and} \quad x < t < x + x(1-x) \\ 0 & \text{if} \quad x^2 < t \leqq x \quad \text{and} \quad x + x(1-x) < t \leqq 1, \end{cases}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

formula (3.5) permits us to give the representation

$$f(x) = \int_0^1 \{f[\alpha_m(x; t)] + \varphi_m(x; t) f''(t)\}\, dt.$$

6. The Remainder $R_{m,n}(f; x, y)$ in the Case of Approximation by the Bernstein Polynomials $B_{m,n}(f; x, y)$. Let $f(x, y)$ be a real function defined on the square $D : 0 \leqq x, t \leqq 1$, let us suppose that it has for $(x, y) \in D$ the partial derivatives f''_{x^2}, f''_{y^2}, $f^{(iv)}_{x^2y^2}$, and that they are R-integrable on D.

We may consider the approximation formula

$$f(x; y) = B_{m,n}(f; x, y) + R_{m,n}(f; x, y), \tag{6.1}$$

where

$$B_{m,n}(f; x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} p_{m,i}(x) p_{n,j}(y) f\left(\frac{i}{m}, \frac{j}{n}\right), \tag{6.2}$$

are Bernstein polynomials of two variables of the (m, n)th degree. The remainder $R_{m,n}(f; x, y)$ is, by definition, the difference between the function $f(x, y)$ and the polynomial $B_{m,n}(f; x, y)$.

7. The Integral Representation of $R_{m,n}(f; x, y)$. In order to obtain an integral expression of the remainder, we might use the result established for functions of one variable. Indeed, owing to formula (3.5), we have

$$f(x, y) = \sum_{i=0}^{m} p_{m,i}(x) f\left(\frac{i}{m}, y\right) + \int_0^1 \varphi_m(x; t) f''_{t^2}(t, y)\, dt, \tag{7.1}$$

where $\varphi_m(x; t)$ is defined in (3.3). Let us now expand $f(i/m, y)$, also using the result of the first part of this paper. We obtain

$$f\left(\frac{i}{m}, y\right) = \sum_{j=0}^{n} p_{n,j}(y) f\left(\frac{i}{m}, \frac{j}{n}\right) + \int_0^1 \psi_n(y; z) f''_{z^2}\left(\frac{i}{m}, z\right) dz,$$

where $\psi_n(y; z)$ has a definition analogous to $\varphi_m(x: t)$.

Thus, (7.1) becomes

$$\begin{aligned} f(x, y) &= \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x) p_{n,j}(y) f\left(\frac{i}{m}, \frac{j}{n}\right) \\ &\quad + \int_0^1 \varphi_m(x; t) f''_{t^2}(t, y)\, dt + \int_0^1 \psi_n(y; z) \sum_{i=0}^{m} p_{m,i}(x) f''_{z^2}\left(\frac{i}{m}, z\right) dz \end{aligned} \tag{7.2}$$

Owing to formula (3.5) we have

$$\sum_{i=0}^{m} p_{m,i}(x) f''_{z^2}\left(\frac{i}{m}, z\right) = f''_{z^2}(x, t) - \int_0^1 \varphi_m(x; t) f^{(iv)}_{t^2z^2}(t, z)\, dt,$$

and formula (7.2) is reduced to formula (6.1) with the following integral expression of the remainder

$$\begin{aligned} R_{m,n}(f; x, y) &= \int_0^1 \varphi_m(x; t) f''_{t^2}(t, y)\, dt + \int_0^1 \psi_n(y; z) f''_{z^2}(x; z)\, dz \\ &\quad - \int_0^1\int_0^1 \varphi_m(x; t)\psi_n(y; z) f^{(iv)}_{t^2z^2}(t, z)\, dt\, dz. \end{aligned} \tag{7.3}$$

8. A Consequence of the Integral Representation of the Remainder $R_{m,n}(f; x, y)$. If we take into account the fact that the functions $\varphi_m(x; t)$ and $\psi_n(y; z)$ do not change sign if x, y, t, and z belong to the interval [0, 1], we may apply the first law of the mean and obtain

$$\begin{aligned} R_{m,n}(f; x, y) &= \mu_{2,0}(f) \int_0^1 \varphi_m(x; t)\, dt + \mu_{0,2}(f) \int_0^1 \psi_n(y; z)\, dz \\ &\quad - \mu_{2,2}(f) \int_0^1\int_0^1 \varphi_m(x; t)\psi_n(y; z)\, dt\, dz, \end{aligned} \tag{8.1}$$

where

$$\inf_D f^{(i+k)}_{x^iy^k}(x, y) \leqq \mu_{i,k}(f) \leqq \sup_D f^{(i+k)}_{x^iy^k}(x, y).$$

Taking into account (5.2) and a formula analogous to this, we obtain the following expression for the remainder of formula (6.1):

$$\begin{aligned} R_{m,n}(f; x, y) &= -\frac{x(1-x)}{2m}\mu_{2,0}(f) - \frac{y(1-y)}{2n}\mu_{0,2}(f) \\ &\quad - \frac{x(1-x)y(1-y)}{4mn}\mu_{2,2}(f). \end{aligned} \tag{8.2}$$

If we suppose that the partial derivatives which appear in (7.3) are continuous on D, we may obtain from (8.2) an expression analogous to that in (5.4), which corresponds to the case of one variable. In order to allow only two unknown numbers ξ and η from the interval [0, 1], we shall proceed in the following way. From (7.3) we have

(8.3)
$$\begin{aligned}
R_{m,n}(f; x, y) &= \int_0^1 \varphi_m(x; t) \left[f''_{t^2}(t, y) - \int_0^1 \psi_m(y; z) f^{(iv)}_{t^2z^2}(t, z)\, dz \right] dt \\
&+ \int_0^1 \psi_n(y; z) f''_{z^2}(x, z)\, dz = \left[f''_{x^2}(\xi, y) - \int_0^1 \psi_n(y; z) f^{(iv)}_{x^2z^2}(\xi, z)\, dz \right] \int_0^1 \varphi_m(x; t)\, dt \\
&+ \int_0^1 \psi_n(y; z) f''_{z^2}(x, z)\, dz = -\frac{x(1-x)}{2m} f''_{x^2}(\xi, y) \\
&+ \frac{x(1-x)}{2m} \int_0^1 \psi_n(y; z) f^{(iv)}_{x^2z^2}(\xi, z)\, dz + \int_0^1 \psi_n(y; z) f''_{z^2}(x, z)\, dz \\
&= -\frac{x(1-x)}{2m} f''_{x^2}(\xi, y) + \int_0^1 \psi_n(y; z) \left[f''_{z^2}(x, z) + \frac{x(1-x)}{2m} f^{(iv)}_{x^2z^2}(\xi, z) \right] dz. \\
&= -\frac{x(1-x)}{2m} f''_{x^2}(\xi, y) + \left[f''_{y^2}(x, \eta) + \frac{x(1-x)}{2m} f^{(iv)}_{x^2y^2}(\xi, y) \right] \int_0^1 \psi_n(y; z)\, dz.
\end{aligned}$$

Finally we obtain the following evaluation:

$$R_{m,n}(f; x, y) = -\frac{x(1-x)}{2m} f''_{x^2}(\xi, y) - \frac{y(1-y)}{2n} f''_{y^2}(x, \eta) - \frac{x(1-x)y(1-y)}{4mn} f^{(iv)}_{x^2y^2}(\xi, \eta),$$

where ξ and η belong to the interval [0, 1] and have the same values in both terms in which each appears.

9. The Expression of $R_{m,n}(f;\, x,\, y)$ by Two-Dimensional Divided Differences. By using the method of Section 7 we may also extend to two variables the following formula, given by O. Aramă [1], for the remainder of formula (2.2).

$$R_m(f; x) = -\frac{x(1-x)}{m} [t_1, t_2, t_3; f], \tag{9.1}$$

where t_1, t_2, t_3 are points of [0, 1] and $[t_1, t_2, t_3; f]$ represents the divided difference of function $f(x)$ on the points t_i $(i = 1, 2, 3)$.

Let us suppose in this case that $f(x, y)$ is continuous on the domain D. Owing to Aramă's formula we have

$$f(x, y) = \sum_{i=0}^{m} p_{m,i}(x) f\left(\frac{i}{m}, y\right) - \frac{x(1-x)}{m} \left[t_1, t_2, t_3; f \right]_x,$$

where t_i are points from [0, 1], while the divided difference refers to variable x. Then

$$f\left(\frac{i}{m}, y\right) = \sum_{j=0}^{n} p_{n,j}(y) f\left(\frac{i}{m}, \frac{j}{n}\right) - \frac{y(1-y)}{n} \left[z_1, z_2, z_3; f\left(\frac{i}{m}, y\right) \right]_y,$$

where z_1, z_2, z_3 are points from [0, 1].

Thus,

$$f(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} p_{m,i}(x) p_{n,j}(y) f\left(\frac{i}{m}, \frac{j}{n}\right)$$

$$- \frac{x(1-x)}{m} [t_1, t_2, t_3; f]_x - \frac{y(1-y)}{n} \sum_{i=0}^{m} p_{m,i}(x) \left[z_1, z_2, z_3; f\left(\frac{i}{m}, y\right)\right]_y.$$

But, on the other hand we have

$$\sum_{i=0}^{m} p_{m,i}(x) \left[z_1, z_2, z_3; f\left(\frac{i}{m}, y\right)\right]_y = \sum_{i=0}^{m} \left[z_1, z_2, z_3; p_{m,i}(x) f\left(\frac{i}{m}, y\right)\right]_y$$

$$= \left[z_1, z_2, z_3; \sum_{i=0}^{m} p_{m,i}(x) f\left(\frac{i}{m}, y\right)\right]_y$$

and

$$\sum_{i=0}^{m} p_{m,i}(x) f\left(\frac{i}{m}, y\right) = f(x, y) + \frac{x(1-x)}{m} [t_1, t_2, t_3; f]_x.$$

If we take into account that

$$[t_1, t_2, t_3; [z_1, z_2, z_3; f]_y]_x = [z_1, z_2, z_3; [t_1, t_2, t_3; f]_x]_y = \begin{bmatrix} t_1, t_2, t_3 \\ z_1, z_2, z_3 \end{bmatrix}; f\Bigg],$$

this being the two-dimensional divided difference, we obtain formula (6.1) with the following expression for the remainder:

$$R_{m,n}(f; x, y) = -\frac{x(1-x)}{m} [t_1, t_2, t_3; f]_x - \frac{y(1-y)}{n} [z_1, z_2, z_3; f]_y$$

(9.2)

$$- \frac{x(1-x)y(1-y)}{mn} \left[\begin{matrix} t_1, t_2, t_3 \\ z_1, z_2, z_3 \end{matrix}; f\right].$$

10. Applications. Using the approximation formulas (2.2) and (6.1), one may construct certain formulas for numerical differentiation and integration of functions of one and two variables.

Thus, for instance, we have the numerical differentiation formulas

$$f'(0) = m\left[f\left(\frac{1}{m}\right) - f(0)\right] - \frac{1}{2m} f''(\xi), \qquad 0 < \xi < m^{-1}$$

$$f''(0) = m(m-1)\left[f\left(\frac{2}{m}\right) - 2f\left(\frac{1}{m}\right) + f(0)\right] - \left(1 - \frac{1}{m}\right) f''(\xi), \qquad 0 < \xi < 2m^{-1}$$

and the numerical integration formulas

$$\int_0^1 f(x)\, dx = \frac{1}{m+1} \sum_{i=0}^{m} f\left(\frac{i}{m}\right) - \frac{1}{12m} f''(\xi),$$

(10.1)

$$\int_0^1 \int_0^1 f(x, y)\, dx\, dy = \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{j=0}^{n} f\left(\frac{i}{m}, \frac{j}{n}\right)$$

$$- \frac{1}{12m} f''_{x^2}(\xi, \eta') - \frac{1}{12n} f''_{y^2}(\xi', \eta) - \frac{1}{144mn} f^{(iv)}_{x^2y^2}(\xi, \eta),$$

where ξ, ξ', η, η' belong to the interval $(0, 1)$.

The quadrature formula (10.1) without an expression for the remainder may be seen in [4].

University of Wisconsin
Madison, Wisconsin, and

University of Cluj
Rumania

1. O. ARAMĂ, "On the properties on monotonicity of the sequence of interpolation polynomials of S. N. Bernstein and their application to the study of approximation of functions," *Mathematica* (Cluj), v. 2 (25), 1960, p. 25–40 (Russian).
2. M. KAC, "Une remarque sur les polynomes de M. S. Bernstein," *Studia Math.*, v. 7, 1938, p. 49–51.
3. G. G. LORENTZ, *Bernstein polynomials*, Toronto, 1953.
4. I. P. NATANSON, *Constructive theory of Functions*, Moscow, 1949 (Russian).
5. D. D. STANCU, "Considerations on polynomial interpolation of functions of several variables," *Bull. Univ. Babes-Bolyai* (Cluj), v. 1, 1957, p. 43–82 (Rumanian).
6. E. VORONOWSKAJA, "Détermination de la forme asymptotique d'approximation des functions par les polynomes de M. Bernstein," *C. R. Acad. Sci. URSS*, *A.*, 1932, p. 79–85. (Russian).
7. DIMITRIE STANCU, "On the remainder in the approximation formalae by Bernstein's polynomials," Abstract 588-2, *Notices*, American Mathematical Society, v. 9, n. 1, 1962, p. 26.