Alternating Direction Iteration Methods For n Space Variables

By J. Douglas, Jr., R. B. Kellogg, and R. S. Varga

We consider the iterative solution of the system of linear equations

$$(1) \qquad (X_1 + X_2 + \cdots + X_n)z = f, \qquad n \geqq 2,$$

where each X_j, $1 \leqq j \leqq n$, is a Hermitian and positive definite $N \times N$ matrix. If $n = 2$, the iterative methods of Peaceman-Rachford [1, Chapter 7], or D'yakonov [2] and Kellogg [3], may be used to solve (1). In this paper these methods are generalized to $n \geqq 2$, and are shown, in a sense, to be dual to one another.

Let $\rho > 0$ be fixed, and define $z_j = (\rho I + X_j)z$. From (1) we get the compound $nN \times nN$ matrix equation

$$(2) \qquad \begin{bmatrix} I & -W_2(\rho) & \cdots & -W_n(\rho) \\ -W_1(\rho) & I & & -W_n(\rho) \\ \vdots & & & \vdots \\ -W_1(\rho) & -W_2(\rho) & \cdots & I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} f \\ f \\ \vdots \\ f \end{bmatrix},$$

where

$$(3) \qquad W_j(\rho) = (\rho I + X_j)^{-1} \left(\frac{\rho}{n-1} I - X_j \right).$$

Our first set of alternating direction iterative methods will be the block Jacobi and block Gauss-Seidel iterative methods applied to (2), namely

$$(4) \qquad (\rho I + X_j)u_j^{(m+1)} = \sum_{k \neq j} \left(\frac{\rho}{n-1} I - X_k \right) u_k^{(m)} + f, \qquad 1 \leqq j \leqq n,$$

and

$$(5) \qquad \begin{aligned} (\rho I + X_j)u_j^{(m+1)} &= \sum_{k<j} \left(\frac{\rho}{n-1} I - X_k \right) u_k^{(m+1)} \\ &\quad + \sum_{k>j} \left(\frac{\rho}{n-1} I - X_k \right) u_k^{(m)} + f. \end{aligned}$$

If $n = 2$, (5) is the Peaceman-Rachford method.

We now form the transpose of the matrix of (2), and consider the compound matrix equation

$$(6) \qquad \begin{bmatrix} I & -W_1(\rho) & \cdots & -W_1(\rho) \\ -W_2(\rho) & I & & -W_2(\rho) \\ \vdots & \vdots & & \vdots \\ -W_n(\rho) & -W_n(\rho) & & I \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{bmatrix}.$$

If the block Jacobi and block Gauss-Seidel iterative methods are applied to (6),

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one obtains the alternating direction iterative methods

$$(7)\qquad (\rho I + X_j)y_j^{(m+1)} = \left(\frac{\rho}{n-1} I - X_j\right)\sum_{k\neq j} y_k^{(m)} + f_j, \qquad 1 \leqq j \leqq n,$$

and

$$(8)\qquad (\rho I + X_j)y_j^{(m+1)} = \left(\frac{\rho}{n-1} I - X_j\right)\{\sum_{k<j} y_k^{(m+1)} + \sum_{k>j} y_k^{(m)}\} + f_j, \qquad 1 \leqq j \leqq n.$$

Here $f_j = (\rho I + X_j)g_j$, and it is assumed that

$$(9)\qquad f_1 + \cdots + f_n = f.$$

When $n = 2$, (8) is the method of D'yakonov. Thus, the Peaceman-Rachford iterative method and D'yakonov's method (and their generalization) are dual to one another in the sense that either can be viewed as the Gauss-Seidel iterative method applied to a particular composite matrix or its transpose.

Since each matrix X_j is Hermitian and positive definite, let the eigenvalues $\lambda_i(j)$ of X_j satisfy

$$0 < a \leqq \lambda_i(j) \leqq b, \qquad 1 \leqq i \leqq N, \qquad 1 \leqq j \leqq n.$$

THEOREM. *If* $\rho > (n-2)b/2$, *and* $\{u_j^{(m)}\}$ *is defined by* (4) *or* (5), *and* $\{y_j^{(m)}\}$ *is defined by* (7) *or* (8), *then*

$$(10)\qquad \lim_{m\to\infty} u_j^{(m)} = z \quad \textit{for each} \quad 1 \leqq j \leqq n,$$

and

$$(11)\qquad \lim_{m\to\infty} (y_1^{(m)} + \cdots + y_n^{(m)}) = z,$$

where z *is the solution of* (1).

Proof. Using spectral (L_2) norms, it is easy to see that there exists a $q < 1$ such that

$$\| W_j(\rho) \| = \max_{1\leqq i\leqq N} \left| \frac{\frac{\rho}{n-1} - \lambda_i(j)}{\rho + \lambda_i(j)} \right| \leqq \frac{q}{n-1} < \frac{1}{n-1}, \qquad 1 \leqq j \leqq n$$

for $\rho > \left(\frac{n-2}{2}\right) b$. Letting

$$\zeta = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

denote a column vector with nN components it is readily verified that the quantity

$$(12)\qquad \| \zeta \| = \max_{1\leqq j\leqq n} \| v_j \|$$

satisfies all the axioms for a vector norm, where in particular we are using Euclidean norms for the subvectors v_i of ζ. Let us denote the compound matrix of (2) by

$I - W$. It follows that for all ζ and all $\rho > \left(\frac{n-2}{2}\right) b$,

$$\| W\zeta \| \leqq q \| \zeta \| < \| \zeta \| , \tag{13}$$

so W is a convergent matrix. But, as W is just the block Jacobi iteration matrix derived from (2), the block Jacobi iterative method of (4) is convergent. If W^* is the conjugate transpose of W, the same argument shows that $\| W^*\zeta \| < \| \zeta \|$ so the iterative method of (7) is also convergent. A similar argument shows that the block Gauss-Seidel methods (5) and (8) are convergent.

If $u_j = \lim_{m\to\infty} u_j^{(m)}$ in (4) or (5), the z_j satisfy the system of equations

$$(\rho I + X_j)u_j = \sum_{k\neq j} \left(\frac{\rho}{n-1} I - X_k\right) u_k + f, \qquad 1 \leqq j \leqq n.$$

Using (13), it may be seen that this system has a unique solution. Since $u_j = z$, $1 \leqq j \leqq n$, is a solution, (10) is obtained.

If $y_j = \lim_{m\to\infty} y_j^{(m)}$ in (7) or (8), the y_j satisfy the system of equations

$$(\rho I + X_j)y_j = \left(\frac{\rho}{n-1} I - X_j\right) \sum_{k\neq j} y_k + f_j, \qquad 1 \leqq j \leqq n.$$

Adding these, one obtains $(X_1 + \cdots + X_n)(y_1 + \cdots + y_n) = f$, so that (11) is obtained, proving the theorem.

We remark that this theorem can also be deduced as an application of a generalization [4] of the well known result of Collatz [5], viz., that a strictly diagonally dominant matrix gives rise to convergent Jacobi and Gauss-Seidel iterative methods. For the norms of (12), the partitioned matrix of (2) or (6) is block strictly diagonally dominant in the sense of [6].

Because of the restriction $\rho > (n - 2)b/2$, it is doubtful that this procedure converges very rapidly, and for this reason, no estimates of rates of convergence are included. (This restriction on ρ is necessary even in the favorable case when the X_j all commute with one another.) We stress, however, that the main point of this paper is the theoretical result of *convergence* without commutativity assumptions on the matrices X_j. To our knowledge, similar results have not been proved for other alternating direction methods applied to n-dimensional problems, $n \geqq 3$. Complementary to this is the fact that three-dimensional matrix problems have been constructed* for which the Douglas-Rachford method [7] and the generalized Peaceman-Rachford method of Douglas [8] each *diverge* for a suitable single positive parameter ρ.

Finally, it is worth mentioning that our generalization of the Peaceman-Rachford iterative method (5) is computationally more attractive than our generalization of the method of D'yakonov, since the latter requires, from (11), more vector storage in practical applications.

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Generation of Permutations by Adjacent Transposition

By Selmer M. Johnson

1. Introduction. In D. H. Lehmer's review [1] of a paper by Mark Wells [2], "Generation of Permutations by Transposition," he states:

> "The author describes a new systematic method of generation permutations with the interesting feature that each permutation is derived from its predecessor by a single interchange of two marks. In more than half the cases the two marks are adjacent."

In the present paper a different method is described in which each permutation is derived from its predecessor by a single interchange of two marks in adjacent positions. Moreover, the rules are extremely simple.

First the method will be described in terms of the marks themselves. Then the method will be restated in terms of positions.

2. The Method in Terms of Marks. Define indices $I(k)$ for $k = 1, 2, \cdots, n$, where $I(k)$ is 0 or 1 according as the permutation on the marks $1, 2, 3, \cdots, k - 1$ is even or odd. By convention, let $I(1) = I(2) = 0$. Define $T(k)$ to be the interchange of the mark k with some smaller mark immediately to the left [right] according as $I(k)$ is 0[1].

Our rules for generating the next permutation are as follows:

1) At each stage apply $T(m)$, where m is the largest mark for which $T(m)$ is defined.
2) Change the indices $I(k)$ for $m < k \leqq n$.

Repeat the cycle of steps on the new permutation, etc.

Note that according to these rules we apply $T(m)$ only when all larger marks are at the extreme left or right of the set of marks $(1, 2, \cdots, m)$ in some order.

It is clear that this method will generate all $n!$ permutations on n marks once and only once. For each fixed permutation on the marks from 1 to k we move the mark $k + 1$ in one direction through every possible position, thus giving each

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