In the absence of definite information about the problem to be considered, it is difficult to obtain an estimate of the total number of iterations required for the last five methods.

For our method, we obtain:

Auxiliary Method	*Multiplications*	*Additions*
I. Elimination	$n^4/12$	$n^4/12$
II. Seidel (one iteration)	$n^3/3$	$n^3/3$
III. Relaxation (one iteration)	$n^3/3$	$n^3/3$
IV. Gradient method (one iteration)	$2n^3/3$	$2n^3/3$
V. Conjugate gradient method, symmetric positive definite matrix (one iteration)	$n^3/3$	$n^3/3$
VI. Conjugate gradient method, general matrix (one iteration)	n^3	n^3

3.6. Numerical Comparison. For a 10×10 determinant, we obtain the following estimates for the number of operations required for the methods discussed:

Method	*Multiplications*	*Additions*
1. Expansion in elements	3.26×10^7	3.63×10^6
2. Expansion in minors	6.23×10^6	
3. Chio's method	330	330
4. Tridiagonal method	1300	660
5. Auxiliary method (by means of solution of linear equations)		
I	1000	1000
II (one iteration)	330	330
III (one iteration)	330	330
IV (one iteration)	660	660
V (one iteration)	330	330
VI (one iteration)	1000	1000

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Coefficients in Quadrature Formulas

By A. H. Stroud

The following result is well known (see, for example, Krylov [2], p. 104, or Szego [3], p. 48):

Theorem 1. *If $w(x)$ is nonnegative throughout the finite or infinite segment $[a, b]$ and if the quadrature formula*

$$\int_a^b w(x)f(x)\,dx \simeq \sum_{i=1}^n A_i f(x_i) \tag{1}$$

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is exact for all polynomials of degree $\leqq 2n - 2$ *then the coefficients* $A_i (i = 1, 2, \cdots, n)$ *are all positive.*

Here the x_i are assumed to be real numbers. The usual method for proving this theorem is to obtain the following representation for the A_i:

$$A_i = \int_a^b w(x) \left[\frac{\omega(x)}{(x - x_i)\omega'(x_i)} \right]^2 dx \qquad (i = 1, 2, \cdots, n)$$

where $\omega(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$. If $w(x)$ changes sign then this representation still holds, but in general we are not able to determine from it how many of the A_i are positive and how many negative.

In Theorem 2 below we give a method for determining the number of positive and negative A_i in (1) for a more general weight function.

In the remainder of this note we assume that $w(x)$ is any function for which the moments

$$c_i = \int_a^b w(x) x^i \, dx \qquad (i = 0, 1, 2, \cdots)$$

are defined and finite and for which the moment matrix

$$C_n = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \vdots & \vdots \\ c_{n-1} & c_n & \cdots & c_{2n-2} \end{bmatrix} \tag{2}$$

is nonsingular.

THEOREM 2. *If* C_n *is congruent (in the terminology of Gantmacher* [1], *p.* 296) *to a diagonal matrix with* p *positive and* q *negative diagonal elements* $(p + q = n)$*—or equivalently, if* C_n *has* p *positive and* q *negative eigenvalues—then a quadrature formula* (1) *which is exact for all polynomials of degree* $\leqq 2n - 2$ *has* p *positive and* q *negative coefficients.*

Proof. The condition that (1) be exact for all polynomials of degree $\leqq 2n - 2$ is equivalent to the equations

$$c_k = A_1 x_1^k + A_2 x_2^k + \cdots + A_n x_n^k \qquad (k = 0, 1, \cdots, 2n - 2).$$

This system of equations may be written as the following matrix equation:

$$X_n^T D_n X_n = C_n$$

where C_n is the symmetric moment matrix (2) and

$$D_n = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix} \qquad X_n = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

If C_n is congruent to a diagonal matrix with p positive and q negative diagonal elements then there is a nonsingular $n \times n$ matrix B for which $B^T C_n B$ is a diagonal matrix of the stated form. Since

$$(X_n B)^T D_n (X_n B) = B^T C_n B$$

then D_n is also congruent to a diagonal matrix of the stated form. From the law of inertia ([1], p. 296–298) D_n also has p positive and q negative elements and the proof is complete.

From this result Theorem 1 follows as a corollary since, if $w(x)$ is nonnegative, C_n is positive definite and therefore congruent to a diagonal matrix with n positive elements.

As a simple example consider a 2-point quadrature formula of the form

$$\int_{-1}^{1} (3 - 5 \mid x \mid) f(x) \, dx \simeq A_1 f(x_1) + A_2 f(x_2). \tag{3}$$

For this weight function the monomial integrals are $c_0 = 1$, $c_1 = 0$, $c_2 = -1/2$, $c_3 = 0$. There are no real values of x_1, x_2 for which (3) can be made exact for $f(x) = 1, x, x^2, x^3$. There are, however, an infinity of such formulas with real x_1, x_2 which are exact for $f(x) = 1, x, x^2$ and Theorem 2 still applies. One such formula is

$$x_1 = \tfrac{1}{2} \qquad x_2 = 1$$
$$A_1 = 2 \qquad A_2 = -1.$$

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A Partition Problem

By M. H. McAndrew

1. Introduction. The following theorem is proved: *Given integers a, b, c, d, each $\geqq 2$, then either there exist integers m, n with $\mid m - n \mid \leqq 1$, a partition of a into m parts of which each part is coprime to b, and a partition of c into n parts, each part coprime to d; or the same conclusion holds with the roles of a and b reversed and the roles of c and d reversed.*

This question arises in the investigation of the minimum length of input strings required to distinguish two partial automata. Elgot and Rutledge [1] deduce an upper bound for the length of such strings and by using the theorem quoted above show that this upper bound can be attained. In Section 4 we demonstrate by an example that the restriction "$a, b, c, d \geqq 2$" cannot be relaxed.

2. Preliminary Lemmas. In the sequel, all variables are to be taken as strictly positive integers.

Lemma 1. *If $l > 1$, $l = \prod_{i=1}^{r} P_i$ where the p_i are distinct primes, and if m is even, then there is an a such that*

$$(a, l) = 1,$$
$$(m - a, l) = 1.$$

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