which are based on Monte Carlo runs of 10,000 periods of operation. The book is a worthy member of Wiley's publications in Operations Research.

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49[L].—GARY D. BERNARD & AKIRA ISHIMARU, *Tables of the Anger and Lommel-Weber Functions*, Technical Report No. 53, AFCRL 796, University of Washington Press, Seattle, 1962, ix + 65 p., 28 cm. Price $2.00.

These important tables result from work on electromagnetic theory. They were computed on an IBM 709 at the Pacific Northwest Research Computing Laboratory of the University of Washington, with support from the Boeing Company, Seattle and the Air Force Cambridge Research Laboratories, Bedford, Mass. The functions tabulated are the Anger functions.

$$J_\nu(x) = \frac{1}{\pi}\int_0^\pi \cos(\nu\theta - x\sin\theta)\,d\theta$$

and the Lommel-Weber functions

$$E_\nu(x) = \frac{1}{\pi}\int_0^\pi \sin(\nu\theta - x\sin\theta)\,d\theta.$$

When ν is an integer n, the Anger function reduces to the Bessel function $J_n(x)$.

Both functions are tabulated to 5D, without differences, for $\nu = -10(0.1)10$, $x = 0(0.1)10$. Tables for negative x are unnecessary, since changing the sign of both ν and x leaves J unchanged and merely changes the sign of E. There are graphs of both functions against ν and contour maps of both functions in the (ν, x) plane. An appendix contains an IBM 709 FORTRAN program for computing the functions.

Previous tables of the Anger functions (other than the Bessel functions for integral ν) are exceedingly slight. Rather more has been done on the Lommel-Weber functions; see FMRC *Index* [1]. The concise and handy tables of Bernard and Ishimaru now establish both functions firmly in the repertoire of numerically available functions.

Precision is stated to be ± 1 in about the last (fifth) decimal place. If this is taken to mean that the tabular values are always within about one final unit of the true values, the statement appears to be true, but does less than justice to the accuracy of the tables. With perfect rounding of the normal kind, tabular values lie within half a final unit of the true values, and it might be thought that the distribution of the rounding errors in the present tables has twice the perfect scatter. As far as one can judge, this is not so.

Only a small fraction of the tabular values can be compared with values already available, but the chief comparisons which are possible have been carried out by the reviewer, in order to test his hypothesis that the vast majority of the tabular values for $\nu \geqq -\frac{1}{2}$ are correctly rounded according to a different convention. This is that positive values are rounded upwards, and negative values are rounded (numerically) downwards; in other words, that in both cases the tabular values are algebraically greater than the true values, and that the part of the tables mentioned ($\nu \geqq -\frac{1}{2}$) has a positive bias of about half a final unit. If the tabular values conformed perfectly to this convention, the part of the tables mentioned would be just as accurate, properly interpreted, as tables perfectly rounded on the normal convention. Actually, the tabular values checked (which may not be representative, having been selected for comparability with values already available) differ from the true values by between 1 and about $1\frac{1}{4}$ final units in a very small percentage of cases; they are thus comparable in accuracy with values in a normally rounded table "imperfect" to the extent of having a very small percentage of rounding errors lying between $\frac{1}{2}$ and about $\frac{3}{4}$ of a final unit. It is because of the positive bias that $E_{\frac{1}{2}}(0)$ and $E_{-\frac{1}{2}}(0)$, which equal $\pm 2/\pi$, appear as $+0.63662$ and -0.63661 respectively; on the normal convention, the latter is one final unit out, the true digits to 8D being 63661 977.

As far as the Anger functions are concerned, the 1111 values of $J_\nu(x)$ for $\nu = 0(1)10$, $x = 0(0.1)10$ have been read against the 10D British Association values [2] of Bessel functions. There are ten cases in which the tabular value differs from the true value by more than one final unit. These are for:

ν	0	0	0	1	3	3	4	4	5	6
x	8.5	9.4	9.9	9.2	9.8	9.9	9.1	9.5	9.4	9.3

The largest of these discrepancies is for $J_3(9.9)$, where the B.A. tables give $+0.03431\ 83264$ and the present tables have $+0.03433$, so that the difference is less than 1.17 final units. This is so slight an excess over unity that it does not seem worth while to set out details for the other nine cases. There are also 32 cases in which the tabular value is correctly rounded by the ordinary convention, instead of by the hypothetical one. Four of these cases are for $x \geqq 8.2$, and the other 28 are for small x, where the rounding of the very small values of J is sometimes to 0.00000 and sometimes to 0.00001; the latter is given for $J_9(0) = 0$. The six 6D values of $J_\nu(\nu)$ for $\nu = 0(0.1)0.5$ given in Brauer & Brauer [3] make possible five additional comparisons, valuable because they are for non-integral ν; the case $J_0(0) = 1$ has already been included above. The five values for $\nu = 0.1(0.1)0.5$ are all positive, and all are rounded upwards in the present tables.

It must be added that the values of $J_{-n}(x)$ given for $n = 2(2)10$ are those given for $J_n(x)$, and for $n = 1(2)9$ are those given for $J_n(x)$ with the signs changed. Thus the bias of the rounding is reversed for odd negative integral ν, but not for even negative integral ν. This shows that the rounding hypothesis being tested would fail if it were extended to include $\nu = -1$ (but it will be seen to be valid for Lommel-Weber functions at $\nu = -\frac{1}{2}$).

As far as the Lommel-Weber functions are concerned, 439 different values of $E\nu(x)$ were read against values of

$$E_\nu(\nu), E_{\nu-1}(\nu); \qquad E_0(x), E_1(x); \qquad E_{\frac{1}{2}}(x), E_{-\frac{1}{2}}(x)$$

given in the British Association Reports for 1923, 1924, and 1925, respectively [4]. All these B.A. values are to 6D, but with indication of halves of a final unit. We omit 24 cases in which the B.A. value ends in an unqualified zero (and hence is useless for determining the hypothetical rounding, just as a final 5 would be useless for determining the normal rounding). In the remaining 415 comparisons, there are five discrepancies greater than one final unit at:

ν	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
x	8.4	9.9	9.0	9.6	9.9

The greatest discrepancy is at $\nu = 1$, $x = 9.9$, where the B.A. value (for $E = -\Omega$) is $-0.251012\frac{1}{2}$, and the present tables have -0.25100, a difference of about $1\frac{1}{4}$ final units. The accuracy of the B.A. tables appears to be excellent, but a 6D table only partially investigated cannot provide quite the check that a good 10D table does. Nevertheless, this provides further reason to think that about one per cent of the Bernard and Ishimaru values differ from the true values by more than one final unit. There are also four cases in which the rounding is correct by the normal rule, instead of by the hypothetical rule. Of the five additional 6D values of $E_\nu(\nu)$ given in Brauer & Brauer [3], that for $\nu = 0.1$ ends in zero and so is useless for testing the hypothesis; those for $\nu = 0.2(0.1)0.5$ are all positive and all rounded upwards in Bernard & Ishimaru, so that they conform to the hypothesis.

The discussion given above is unavoidably partial and incomplete for $\nu \geqq -\frac{1}{2}$, while for $\nu < -\frac{1}{2}$ it merely shows that the hypothesis needs modifying, without discovering how it should be modified. One would welcome some statement by the authors on a subject which might on occasion be of great interest to users of the tables. Lacking information, users will presumably have either to delve into analytical details and the FORTRAN program, or to accept rounding uncertainties of a size which an authoritative statement might almost halve. The work involved in the discussion, tentative as it is, has been felt to be worthwhile, because those who are interested in special higher mathematical functions are likely to rank the tables of Bernard and Ishimaru among the most important produced in that field since automatic computers began to contribute.

A. F.

1. A. Fletcher, J. C. P. Miller, L. Rosenhead &. L. J. Comrie, *An Index of Mathematical Tables*, second ed., vol. 1, 1962, p. 458. Blackwell, Oxford, England (for scientific Computing Service, London); American ed., Addison-Wesley.
2. British Association for the Advancement of Science, *Mathematical Tables*, vol. 10, 1952, p. 180, Cambridge Univ. Press.
3. P. Brauer & E. Brauer, *Z. Angew. Math. Mech.*, vol. 21, 1941, p. 177–182, especially p. 180–181.
4. British Association for the Advancement of Science, *Reports* for 1923, p. 293; for 1924, p. 280; for 1925, p. 244. London.

50[L].—E. Paran & B. J. Kagle, *Tables of Legendre Polynomials of the First and Second Kind*, Research Report 62-129-103-R1, Westinghouse Research Laboratories, Pittsburgh, Pennsylvania, Sept. 7, 1962, i + 202 p., 28 cm.

These are tables of the functions $P_n(x)$ and $Q_n(x)$, in the usual notation. Since $P_n(x)$ and $Q_n(x) - P_n(x) \tanh^{-1} x$ are polynomials, $Q_n(x)$ is not, so that the word "functions" could profitably replace "polynomials" in the title. $P_n(x)$ is tabulated for $x = 0.001(0.001)1$ and $Q_n(x)$ for $x = 0.001(0.001)0.999$, in both cases for $n = 0(1)27$ and to 6S without differences. At $x = 1$, $Q_n(x)$ is infinite, and the numbers given in the tables are limiting values of $Q_n(x) - P_n(x) \tanh^{-1}x$, although this is not explained in the very brief accompanying text. It is a pity that the