# Quadrature Formulas with Simple Gaussian Nodes and Multiple Fixed Nodes

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**1. Introduction.** Let us consider a definite integral of the form

$$I(w;f) \equiv I(f) = \int_a^b w(x)f(x)\,dx$$

where $f(x)$ is an integrable function on the finite or infinite interval $(a\ b)$, and $w(x)$ is a given, fixed function such that its moments $c_k = I(w;x^k)\,(k = 0, 1, 2, \cdots)$ exist and $c_0 > 0$.

In this paper we consider quadrature formulas which use multiple nodes chosen in advance and other simple nodes which we choose to increase the degree of exactness of the formulas.

To be more precise, let $a_1, a_2, \cdots, a_p$ be real numbers, which are assumed to be fixed, such that the polynomial

$$A(x) = C(x - a_1)^{m_1}(x - a_2)^{m_2} \cdots (x - a_p)^{m_p} \qquad (C \neq 0)$$

where the $m_j$ are positive integers, is nonnegative for all $x$ in $(a, b)$.

Let $x_1, x_2, \cdots, x_n$ be distinct real numbers.

In the following we will assume that $f(x)$ has a derivative of order $m_i - 1$ at the point $a_i (i = 1, \cdots, p)$ and we consider a quadrature formula of the form

$$I(w;f) = V(f) + R(f) \tag{1}$$

where

$$V(f) = \sum_{i=1}^{n} A_i f(x_i) + \sum_{k=1}^{p} \sum_{h=0}^{m_k-1} B_k^{(h)} f^{(h)}(a_k)$$

and the remainder $R(f)$ is then, by definition, the difference $I(w;f) - V(f)$.

Given the nodes $a_k$ and their multiplicities $m_k$ the problem is then to determine the simple nodes $x_i$ and the coefficients $A_i$ and $B_k^{(h)}$ so that formula (1) has the highest degree of exactness. (As usual, we say that (1) has a degree of exactness $s$ if $R(1) = R(x) = \cdots = R(x^s) = 0$ and $R(x^{s+1}) \neq 0$.) We will call the $a_k$ the *fixed nodes* and the $x_i$ *Gaussian nodes*.

In Section 2 we give a vew properties of formula (1) and in Section 3 a brief historical summary of special cases of this formula. In Section 4 we tabulate some particular formulas.

**2. Properties of the Formulas.** The following result is known [19]: The maximum degree of exactness of (1) is $N = 2n + m - 1$, where $m = m_1 + m_2 + \cdots + m_p$, and this is achieved if and only if the polynomial

$$P_n(x) = (x - x_1)(x - x_2) \cdots (x - x_n)$$

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is orthogonal on the interval $(a, b)$ with respect to the weight function $w(x)A(x)$ to any polynomial $U_{n-1}(x)$ of degree $\leqq n - 1$.

Since $P_n(x)$ is orthogonal on $(a, b)$ with respect to a nonnegative weight function it is well known that its roots $x_1, x_2, \cdots, x_n$ are real, distinct and lie in the interior of $(a, b)$. If $a = -b$ so that the midpoint of $(a, b) \equiv (-b, b)$ is the origin and if the weight function $w(x)A(x)$ is even then the roots of $P_n(x)$ will also be symmetric with respect to the origin.

The Gaussian nodes $x_i$ can be found either from a relationship of Christoffel type [19, 23], or by determining the minimum of the following function of $n$ variables

$$F(t_1, \cdots, t_n) = \int_a^b w(x)A(x)(x - t_1)^2 \cdots (x - t_n)^2 \, dx,$$

or by calculating the orthogonal polynomial $P_n(x)$ and its roots by a direct method.

If we use the Lagrange-Hermite interpolating polynomial for $f(x)$ at the nodes $x_i$ and $a_k$, we obtain the following expressions for the coefficients of (1):

$$A_i = I(\phi_i(x)), \qquad B_k^{(h)} = I(\psi_{k,h}(x)) \tag{2}$$

where

$$\phi_i(x) = \frac{A(x)l_i(x)}{A(x_i)l_i(x_i)}, \qquad \psi_{k,h}(x) = \frac{(x - a_k)^h}{h!} \sum_{j=0}^{m_k-h-1} \left[ \frac{(x - a_k)^j}{j!} \left( \frac{1}{v_k(x)} \right)_{a_k}^{(j)} \right] v_k(x)$$

and

$$l_i(x) = \frac{P_n(x)}{x - x_i}, \qquad v_k(x) = \frac{A(x)}{(x - a_k)^{m_k}}.$$

We also have [18]

$$A_i = I\left( \frac{A(x)l_i^2(x)}{A(x_i)l_i^2(x_i)} \right)$$

so that the $A_i$ are all positive. The $B_k^{(h)}$ are not necessarily positive.

The remainder $R(f)$ can be expressed as

$$R(f) = \frac{C^{-1}}{(2n + m)!} f^{(2n+m)}(\xi) I(A(x)P_n^2(x)) = K_{2n+m} f^{(2n+m)}(\xi) \tag{3}$$

where $\xi$ is some point belonging to the smallest segment containing the $x_i$ and $a_k$. This relationship can be derived by considering the interpolation formula

$$f(x) = L \begin{pmatrix} a_1 \; a_2 \cdots a_p \; x_1 \; x_2 \cdots x_n \\ m_1 \; m_2 \cdots m_p \; 2 \;\; 2 \cdots \; 2 \end{pmatrix} f \,\Big|\, x \Bigg) + \rho(f; x)$$

where

$$\rho(f; x) = C^{-1}A(x)P_n^2(x) \begin{bmatrix} a_1 \cdots a_p \; x_1 \cdots x_n \; x \\ m_1 \cdots m_p \; 2 \cdots 2 \;\; 1 \end{bmatrix}; f \Bigg]$$

$$= \frac{C^{-1}}{(2n + m)!} A(x)P_n^2(x) f^{(2n+m)}(\eta)$$

and where the quantity in square brackets is the divided difference of $f(x)$ on the indicated nodes (the numbers beneath the nodes designate their multiplicities). The point $\eta$ belongs to the smallest interval which contains the points $x$, $x_i$ and $a_k$ .

We now mention two particular cases of formula (1):

1. $(a, b) = (-b, b)$, $-a_1 = a_2 = b(p = 2)$; $w(x)$ an even function and $C = 1$. The formula then becomes

$$\begin{aligned}\int_{-b}^{b} w(x)f(x)\,dx &= \sum_{i=1}^{n} A_i f(x_i) + \sum_{j=0}^{m_1-1} B_j f^{(j)}(-b) + \sum_{k=0}^{m_2-1} B_k^* f^{(k)}(b) \\ &+ \frac{f^{(2n+m_1+m_2)}(\xi)}{(2n+m_1+m_2)!}\int_{-b}^{b} w(x)(x+b)^{m_1}(x-b)^{m_2}P_n^2(x)\,dx.\end{aligned} \tag{4}$$

Using (2) one can see that when $m_1 = m_2 = \mu$ we have $B_j = B_j^*$ if $j$ is even and $B_j = -B_j^*$ if $j$ is odd. In this case (4) becomes

$$\begin{aligned}\int_{-b}^{b} w(x)f(x)\,dx &= \sum_{i=1}^{n} A_i f(x_i) + \sum_{j=0}^{\mu-1} B_j[f^{(j)}(-b) + (-1)^j f^{(j)}(b)] \\ &+ \frac{f^{(2n+2\mu)}(\xi)}{(2n+2\mu)!}\int_{-b}^{b} w(x)(x^2-b^2)^{\mu}P_n^2(x)\,dx\end{aligned} \tag{5}$$

where $B_j > 0(j = 0, 1, \cdots, -1)$.

2. $(a, b) = (-b, b)$, $-a_1 = a_3 = b$, $a_2 = 0$, $m_1 = m_3 = \mu$, $m_2 = 2\nu$, $n = 2k$; $w(x)$ an even function; $C = 1$. We then have

$$\begin{aligned}\int_{-b}^{b} w(x)f(w)\,dx &= \sum_{i=1}^{2k} A_i f(x_i) + \sum_{j=0}^{\mu-1} B_j[f^{(j)}(-b) + (-1)^j f^{(j)}(b)] \\ &+ \sum_{j=0}^{\nu-1} C_{2j} f^{(2j)}(0) + \frac{f^{(4k+2\mu+2\nu)}(\xi)}{(4k+2\mu+2\nu)!}\int_{-b}^{b} w(x)(x^2-b^2)^{\mu}x^{2\nu}P_{2k}^2(x)\,dx.\end{aligned} \tag{6}$$

Here the coefficients of $f^{(j)}(0)$ are zero for $j$ odd. If the number of Gaussian nodes $n$ is odd $(2k + 1)$ then one of these will coincide with the origin and a formula similar to (6) is obtained with the multiplicity of the fixed node $a_2 = 0$ increased by 2.

**3. Historical Summary.** Here we mention some classical special cases of formula (1).

1. The case $m = 0$ (no fixed nodes); $(a, b)$ finite; $w(x) \equiv 1$. This corresponds to the classical Gaussian quadrature formula [3]. The $x_i$ are the zeros of the $n$th degree Legendre polynomial corresponding to the segment $(a, b)$.

Stieltjes [20] studied the case for a nonnegative weight function $w(x)$. Mehler [10], Posse [13], Heine [4] and Deruyts [2] constructed formulas of this type for special weight functions. The expression (3) for the remainder was found by Markoff [9].

2. $(a, b)$ finite, $a_1 = a$, $a_2 = b$, $m_1 = m_2 = 1(p = 2)$. Scarborough [16] attributes this formula to Lobatto [8].

3. $(a, b)$ finite, $a_1 = a$, $m_1 = 1(p = 1)$. This formula (as well as that in 2) was studied by Radau [14].

4. $(a, b)$ finite, $w(x) \equiv 1$, simple fixed nodes $a_k$ not interior to $(a, b)$. This case was studied by Christoffel [1] who also gave a famous formula for finding the Gaussian nodes in this case. One derivation of the above expression for the remainder was given in [18]; see also [7].

5. $(a, b)$ finite, $w(x) \equiv 1$, $a_1 = a$, $a_2 = b (p = 2)$; no Gaussian nodes. Here

$$\int_a^b f(x)\,dx = \sum_{i=0}^{m_1-1} \frac{(b-a)^{i+1}}{(i+1)!} \frac{C_{m_1,i+1}}{C_{m_1+m_2,i+1}} f^{(i)}(a)$$

$$- \sum_{j=0}^{m_2-1} \frac{(a-b)^{j+1}}{(j+1)!} \frac{C_{m_2,j+1}}{C_{m_1+m_2,j+1}} f^{(j)}(b) + \frac{(-1)^{m_1} m_1! m_2! (b-a)^{m_1+m_2+1}}{(m_1+m_2)!(m_1+m_2+1)!} f^{(m_1+m_2)}(\xi)$$

where $C_{n,k} = \dfrac{n!}{k!(n-k)!}$. This formula was first given by Hermite [5]. It was also derived by other methods by Obreschkoff [11] and Stancu [18]. The special case $m_1 = m_2$ was given (without proof) by Petr [12].

For additional discussions of some of the above mentioned formulas see Ionescu [6] and Krylov [7]. For references to tables giving nodes and coefficients in several of these formulas see [21].

**4. Numerical Results.** In the accompanying tables we give numerical values of the nodes and coefficients in certain of the above formulas. The formulas are divided in the following way:

Table 1. $\int_{-1}^{1} f(x)\,dx.$

Table 2. $\int_{-\infty}^{\infty} e^{-x^2} f(x)\,dx.$

Table 3. $\int_{0}^{\infty} e^{-x} f(x)\,dx.$

The formulas of Table 1 have fixed nodes at the ends and/or the middle of the interval $[-1, 1]$. The formulas of Tables 2 and 3 have a single fixed node at $x = 0$. The formulas of Tables 1 and 2 are all symmetric with respect to the origin and we tabulate only the nodes and coefficients for $x_i \geqq 0$, $a_k \geqq 0$. As mentioned above the coefficients of the terms $f^{(j)}(-1)$ and $f^{(j)}(1)$, for $j$ odd, in the formulas of Table 1, have opposite signs.

As also previously mentioned, the symmetric formulas in which the point $x = 0$ is a fixed node have the property that the coefficients of the odd derivatives $f^{(1)}(0)$, $f^{(3)}(0)$, $\cdots$, are zero. In such a case if $x = 0$ is the only fixed node, and if it has multiplicity $2\nu$, then the formulas of this type, in Table 1 for example, become

$$\int_{-1}^{1} f(x)\,dx \simeq \sum_{i=1}^{2k} A_i f(x_i) + \sum_{j=0}^{\nu-1} B_{2j} f^{(2j)}(0). \tag{7}$$

These formulas share the property of the classical Gaussian formulas that a formula with $N = 2k + \nu$ terms is exact for all polynomials of degree $\leqq 2N - 1$.

For the multiple nodes the first coefficient given in the tables is the coefficient of the value of the function at the node, the second is the coefficient of the first derivative, etc. For the symmetric formulas with a fixed node $x = 0$, the coefficients of the odd derivatives $f^{(1)}(0)$, $f^{(3)}(0)$, $\cdots$, are omitted since these are zero.

TABLE 1.

$$\int_{-1}^{1} f(x)dx$$

| Node | | Coeff. |
|---|---|---|
| | Fixed node 0<br>Multiplicity 4 | |
| $n = 2$ | $K_8 = (-6)0.4499$ | |
| 0.8451542547 2851657751 | | 0.3920000000 0000000000 |
| 0.0000000000 0000000000 | | (1)0.1216000000 0000000000 |
| 0.0000000000 0000000000 | | (−1)0.5333333333 3333333333 |
| $n = 4$ | $K_{12} = (-11)0.2097$ | |
| 0.9290483037 5689950193 | | 0.1803531769 6630636317 |
| 0.6399972828 1743550078 | | 0.3912386597 6838751438 |
| 0.0000000000 0000000000 | | 0.8568163265 3061224490 |
| 0.0000000000 0000000000 | | (−1)0.1741496598 6394557823 |
| $n = 6$ | $K_{16} = (-17)0.2816$ | |
| 0.9591472977 2322447741 | | 0.1042050388 7776484136 |
| 0.7901728520 6150630704 | | 0.2305465186 0949285501 |
| 0.5056316101 0287030224 | | 0.3355250865 0367200884 |
| 0.0000000000 0000000000 | | 0.6594467120 1814058957 |
| 0.0000000000 0000000000 | | (−2)0.7739984882 8420256992 |
| $n = 8$ | $K_{20} = (-23)0.1455$ | |
| 0.9734211872 3582612217 | | (−1)0.6792870038 2682210895 |
| 0.8623889137 5458573927 | | 0.1526688383 2138161914 |
| 0.6720868083 5941206126 | | 0.2255607194 5999235514 |
| 0.4157226832 7221438146 | | 0.2861022317 0682329609 |
| 0.0000000000 0000000000 | | 0.5354790202 5824103746 |
| 0.0000000000 0000000000 | | (−2)0.4093876301 6685094607 |
| | Fixed node 0<br>Multiplicity 6 | |
| $n = 2$ | $K_{10} = (-8)0.2474$ | |
| 0.8819171036 8819686350 | | 0.3036234902 1241149521 |
| 0.0000000000 0000000000 | | (1)0.1392753019 5751770096 |
| 0.0000000000 0000000000 | | (−1)0.9718172983 4791059281 |
| 0.0000000000 0000000000 | | (−2)0.1360544217 6870748299 |
| $n = 4$ | $K_{14} = (-14)0.4787$ | |
| 0.9429254231 1628495845 | | 0.1457629708 3110661854 |
| 0.7039226030 2987827776 | | 0.3321317563 8874747746 |
| 0.0000000000 0000000000 | | (1)0.1044210545 5602918080 |
| 0.0000000000 0000000000 | | (−1)0.3916063780 0093582407 |
| 0.0000000000 0000000000 | | (−3)0.2687494750 9868144789 |
| $n = 6$ | $K_{18} = (-20)0.3447$ | |
| 0.9658711834 8440782743 | | (−1)0.8722988880 9221794608 |
| 0.8231248936 7021267276 | | 0.1966845031 5952765048 |
| 0.5750954154 9473837175 | | 0.3001707516 1811690881 |
| 0.0000000000 0000000000 | | 0.8318297128 2626729221 |
| 0.0000000000 0000000000 | | (−1)0.1941849808 2691032598 |
| 0.0000000000 0000000000 | | (−4)0.7995852151 6963075405 |
| | Fixed nodes −1, 1<br>Multiplicities 2, 2 | |
| $n = 2$ | $K_8 = (-6)0.7199$ | |
| 0.3779644730 0922722721 | | 0.7259259259 2592592593 |

TABLE 1. (*Continued*)

| Node | Coeff. |
|---|---|
| 1.0000000000 0000000000 | 0.2740740740 7407407407 |
| 1.0000000000 0000000000 | −(−1)0.2222222222 2222222222 |
| $n = 3$ | $K_{10} =$ (−8)0.1697 |
| 0.5773502691 8962576451 | 0.5142857142 8571428571 |
| 0.0000000000 0000000000 | 0.6095238095 2380952381 |
| 1.0000000000 0000000000 | 0.1809523809 5238095238 |
| 1.0000000000 0000000000 | −(−2)0.9523809523 8095238095 |
| $n = 4$ | $K_{12} =$ (−11)0.2876 |
| 0.6947465906 0686574510 | 0.3800412240 4242894414 |
| 0.2505628070 8573158101 | 0.4913873473 8614248443 |
| 1.0000000000 0000000000 | 0.1285714285 7142857143 |
| 1.0000000000 0000000000 | −(−2)0.4761904761 9047619048 |
| $n = 5$ | $K_{14} =$ (−14)0.3647 |
| 0.7694553243 3178732702 | 0.2910651906 0725028794 |
| 0.4209148050 2381144473 | 0.3960953032 1991020589 |
| 0.0000000000 0000000000 | 0.4334391534 3915343915 |
| 1.0000000000 0000000000 | (−1)0.9611992945 3262786596 |
| 1.0000000000 0000000000 | −(−2)0.2645502645 5026455026 |

Fixed nodes −1, 1
Multiplicities 3, 3

| Node | Coeff. |
|---|---|
| $n = 2$ | $K_{10} =$ (−8)0.4524 |
| 0.3333333333 3333333333 | 0.6508928571 4285714286 |
| 1.0000000000 0000000000 | 0.3491071428 5714285714 |
| 1.0000000000 0000000000 | −(−1)0.4642857142 8571428571 |
| 1.0000000000 0000000000 | (−2)0.2380952380 9523809524 |
| $n = 3$ | $K_{12} =$ (−11)0.6472 |
| 0.5222329678 6709351453 | 0.4841600529 1005291005 |
| 0.0000000000 0000000000 | 0.5417989417 9894179894 |
| 1.0000000000 0000000000 | 0.2449404761 9047619048 |
| 1.0000000000 0000000000 | −(−1)0.2261904761 9047619048 |
| 1.0000000000 0000000000 | (−3)0.7936507936 5079365079 |
| $n = 4$ | $K_{14} =$ (−14)0.7294 |
| 0.6406425159 6974405186 | 0.3724625377 7830420877 |
| 0.2260876561 6551863388 | 0.4457279384 1217198170 |
| 1.0000000000 0000000000 | 0.1818095238 0952380952 |
| 1.0000000000 0000000000 | −(−1)0.1238095238 0952380952 |
| 1.0000000000 0000000000 | (−3)0.3174603174 6031746032 |

Fixed nodes −1, 0, 1
Multiplicities 1, 4, 1

| Node | Coeff. |
|---|---|
| $n = 2$ | $K_{10} =$ (−8)0.1414 |
| 0.7453559924 9992989880 | 0.4165714285 7142857143 |
| 0.0000000000 0000000000 | (1)0.1024000000 0000000000 |
| 0.0000000000 0000000000 | (−1)0.3047619047 6190476190 |
| 1.0000000000 0000000000 | (−1)0.7142857142 8571428571 |
| $n = 4$ | $K_{14} =$ (−14)0.3191 |
| 0.8666201864 7293631106 | 0.2198172764 5940025110 |
| 0.5708699758 4449124639 | 0.3669824211 9744026289 |
| 0.0000000000 0000000000 | 0.7523265306 1224489796 |
| 0.0000000000 0000000000 | (−1)0.1160997732 4263038549 |
| 1.0000000000 0000000000 | (−1)0.3703703703 7037037037 |

TABLE 1. (*Continued*)

| Node | | Coeff. |
|---|---|---|
| | $n = 6 \quad K_{18} = (-20)0.2507$ | |
| 0.9176224614 5004323870 | | 0.1366219928 5366296019 |
| 0.7317425457 6912532453 | | 0.2322342793 1505292856 |
| 0.4590829544 5521662140 | | 0.3112010356 0287759713 |
| 0.0000000000 0000000000 | | 0.5944308390 0226757370 |
| 0.0000000000 0000000000 | | (−2)0.5629079914 7942005085 |
| 1.0000000000 0000000000 | | (−1)0.2272727272 7272727273 |
| | Fixed nodes −1, 0, 1<br>Multiplicities 2, 4, 2 | |
| | $n = 2 \quad K_{12} = (-11)0.4045$ | |
| 0.6741998624 6324208625 | | 0.4131499118 1657848325 |
| 0.0000000000 0000000000 | | 0.9020952380 9523809524 |
| 0.0000000000 0000000000 | | (−1)0.2031746031 7460317460 |
| 1.0000000000 0000000000 | | 0.1358024691 3580246914 |
| 1.0000000000 0000000000 | | −(−2)0.5291005291 0052910053 |
| | $n = 4 \quad K_{16} = (-17)0.4589$ | |
| 0.8138467317 9583272311 | | 0.2381579895 5781909663 |
| 0.5205639542 6361903593 | | 0.3455285813 6684706613 |
| 0.0000000000 0000000000 | | 0.6788670377 2418058132 |
| 0.0000000000 0000000000 | | (−2)0.8443619872 1913007627 |
| 1.0000000000 0000000000 | | (−1)0.7687991021 3243546577 |
| 1.0000000000 0000000000 | | −(−2)0.1683501683 5016835017 |
| | $n = 6 \quad K_{20} = (-23)0.2149$ | |
| 0.8789930878 7848235622 | | 0.1561028370 3315345276 |
| 0.6848391660 6458169225 | | 0.2299263327 2268234839 |
| 0.4235789178 7099869992 | | 0.2915266077 9994175462 |
| 0.0000000000 0000000000 | | 0.5455877455 8774558775 |
| 0.0000000000 0000000000 | | (−2)0.4330061472 9186157758 |
| 1.0000000000 0000000000 | | (−1)0.4965034965 0349650350 |
| 1.0000000000 0000000000 | | −(−3)0.6993006993 0069930070 |

TABLE 2.

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\, dx$$

| Node | Coeff. |
|---|---|
| Fixed node 0<br>Multiplicity 4 | |
| $n = 2$ | $K_8 = (-4)0.8242$ |
| (1)0.1581138830 0841896660 | 0.1063472310 5433096164 |
| 0.0000000000 0000000000 | (1)0.1559759388 7968541040 |
| 0.0000000000 0000000000 | 0.1772453850 9055160273 |
| $n = 4$ | $K_{12} = (-7)0.4857$ |
| (1)0.2317504842 1496276763 | (−2)0.5362808829 4294820832 |
| (1)0.1276389950 8430130645 | 0.1921391917 0004230382 |
| 0.0000000000 0000000000 | (1)0.1377449849 8465724555 |
| 0.0000000000 0000000000 | 0.1012830771 9460091585 |
| $n = 6$ | $K_{16} = (-10)0.1501$ |
| (1)0.2910449614 9176071972 | (−3)0.2261464497 1363499464 |
| (1)0.1951635396 6525274434 | (−1)0.1951563938 5124903406 |
| (1)0.1104718207 3084899827 | 0.2452822661 5770052473 |
| 0.0000000000 0000000000 | (1)0.1242405746 9204379010 |
| 0.0000000000 0000000000 | (−1)0.6752205146 3067277230 |
| $n = 8$ | $K_{20} = (-14)0.2840$ |
| (1)0.3422252897 4066300873 | (−5)0.8439217345 3693514448 |
| (1)0.2513821890 6471071714 | (−2)0.1489333585 6034634927 |
| (1)0.1729270753 8701392753 | (−1)0.3786510542 2905663288 |
| 0.9891952628 8025113394 | 0.2772234676 1120866961 |
| 0.0000000000 0000000000 | (1)0.1139281159 2313896958 |
| 0.0000000000 0000000000 | (−1)0.4910694651 8594383440 |
| Fixed node 0<br>Multiplicity 6 | |
| $n = 2$ | $K_{10} = (-5)0.3205$ |
| (1)0.1870828693 3869706928 | (−1)0.3875627953 8750350451 |
| 0.0000000000 0000000000 | (1)0.1694941291 8280153264 |
| 0.0000000000 0000000000 | 0.3074664843 4075278025 |
| 0.0000000000 0000000000 | (−1)0.1582548081 1656393101 |
| $n = 4$ | $K_{14} = (-8)0.1201$ |
| (1)0.2573192636 3099290070 | (−2)0.1512884809 0746850974 |
| (1)0.1542296876 8821252550 | (−1)0.9083232446 2268742521 |
| 0.0000000000 0000000000 | (1)0.1587763432 3628291721 |
| 0.0000000000 0000000000 | 0.2170351654 1700196253 |
| 0.0000000000 0000000000 | (−2)0.7033547027 4028413782 |
| $n = 6$ | $K_{18} = (-12)0.2698$ |
| (1)0.3140292019 9300618726 | (−4)0.5549440850 0930198713 |
| (1)0.2194025119 8397201434 | (−2)0.7066307542 1854978039 |
| (1)0.1350858913 0904357881 | 0.1349845563 5806485698 |
| 0.0000000000 0000000000 | (1)0.1488241134 2880134573 |
| 0.0000000000 0000000000 | 0.1622283054 6321358815 |
| 0.0000000000 0000000000 | (−2)0.3836480196 7651862063 |

TABLE 3.

$$\int_0^\infty e^{-x} f(x)\, dx$$

| Node | Coeff. |
|---|---|
| Fixed node 0, Multiplicity 1 | |
| $n = 2$ | $K_5 = 0.1000$ |
| (1)0.4732050807 5688772935 | (−1)0.4465819873 8520451079 |
| (1)0.1267949192 4311227065 | 0.6220084679 2814621559 |
| 0.0000000000 0000000000 | 0.3333333333 3333333333 |
| $n = 3$ | $K_7 = (-1)0.2857$ |
| (1)0.7758770483 1436335362 | (−2)0.2590933677 1469482431 |
| (1)0.3305407289 3322786046 | 0.1183563854 5510051414 |
| 0.9358222275 2408785919 | 0.6290526808 6775253761 |
| 0.0000000000 0000000000 | 0.2500000000 0000000000 |
| $n = 4$ | $K_9 = (-2)0.7937$ |
| (2)0.1095389431 2683190455 | (−3)0.1201261988 4232922333 |
| (1)0.5731178751 6890996342 | (−1)0.1294284962 0453798249 |
| (1)0.2571635007 6462784750 | 0.1857323340 7684495087 |
| 0.7432919279 8143143546 | 0.6012046901 0385892166 |
| 0.0000000000 0000000000 | 0.2000000000 0000000000 |
| $n = 5$ | $K_{11} = (-1)0.2165$ |
| (2)0.1426010306 5920830849 | (−5)0.4836804002 5232746746 |
| (1)0.8399066971 2048421905 | (−2)0.1038197820 7811716012 |
| (1)0.4610833151 0175324137 | (−1)0.3056192121 4471794526 |
| (1)0.2112965958 5785241511 | 0.2377135666 0681701385 |
| 0.6170308532 7827039571 | 0.5640148108 8726083008 |
| 0.0000000000 0000000000 | 0.1666666666 6666666667 |
| Fixed node 0, Multiplicity 2 | |
| $n = 2$ | $K_6 = (-1)0.6667$ |
| (1)0.6000000000 0000000000 | (−1)0.1388888888 8888888889 |
| (1)0.2000000000 0000000000 | 0.3750000000 0000000000 |
| 0.0000000000 0000000000 | 0.6111111111 1111111111 |
| 0.0000000000 0000000000 | 0.1666666666 6666666667 |
| $n = 3$ | $K_8 = (-1)0.1786$ |
| (1)0.9171029785 6030672021 | (−3)0.6747892865 7219297605 |
| (1)0.4311583133 7195203019 | (−1)0.4872308920 9340772836 |
| (1)0.1517387080 6774124950 | 0.4506021215 0408703419 |
| 0.0000000000 0000000000 | 0.5000000000 0000000000 |
| 0.0000000000 0000000000 | 0.1000000000 0000000000 |
| $n = 4$ | $K_{10} = (-2)0.4762$ |
| (2)0.1245803677 1951138559 | (−4)0.2805737174 0419290005 |
| (1)0.6902692605 8516133972 | (−2)0.4338057261 4037032947 |
| (1)0.3412507358 6969459701 | (−1)0.9131862920 8517077505 |
| (1)0.1226763263 5003020735 | 0.4820930339 3611657769 |
| 0.0000000000 0000000000 | 0.4222222222 2222222222 |
| 0.0000000000 0000000000 | (−1)0.6666666666 6666666667 |

It should be noted that the Gaussian nodes $x_i$ in (7) coincide with the nodes in the quadrature formulas of the form

$$\int_{-1}^{1} x^{2n} f(x)\, dx \simeq \sum_{i=1}^{m} A_i f(x_i)$$

discussed by Rothmann [15].

Each formula was computed by calculating the appropriate orthogonal polynomial for the Gaussian nodes from the recursion relation:

$$P_n(x) = [x - b_n]P_{n-1}(x) - c_n P_{n-2}(x)$$
$$P_{-1}(x) \equiv 0, \qquad P_0(x) \equiv 1.$$

The coefficients $b_n$, $c_n$ in this relation were computed from

$$b_n = \frac{I(w; x^n P_{n-1})}{I(w; x^{n-1} P_{n-1})} + \beta_{n-1}$$
$$c_n = \frac{I(w; x^{n-1} P_{n-1})}{I(w; x^{n-1} P_{n-2})}$$

where $\beta_{n-1}$ is the coefficient of $x^{n-2}$ in $P_{n-1}(x)$:

$$P_{n-1}(x) = x^{n-1} + \beta_{n-1} x^{n-2} + \cdots .$$

The zeros of $P_n(x)$ were found by the Newton-Raphson iterative method. The coefficients in the quadrature formula were found by solving a linear system of equations. The computations were all carried out in double precision using the multiple precision floating point program described in [22]; in double precision this program carries about 24 significant figures. The computations were checked computing all the monomial integrals for which a given formula should be exact and comparing these with the exact values. The values given in the tables are all exact to within one unit in the 20th significant figure.

For each formula given in the tables $K_{2n+m}$ is the constant in the remainder representation of equation (3); here $n$ is the number of Gaussian nodes, $m$ is the sum of the multiplicities of all the fixed nodes and the formula is exact for all polynomials of degree $<2n + m$. A number in parentheses in front of a node or coefficient is the power of ten by which the fractional part must be multiplied to obtain the true number.

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