Jacobi Polynomial Expansions of a Generalized Hypergeometric Function over a Semi-Infinite Ray

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1. Introduction. Suppose $f(x)$ is continuous and has a piecewise continuous derivative for $0 \leqq x/\lambda \leqq 1$. Then $f(x)$ may be expanded into a uniformly convergent series of shifted Jacobi polynomials in the form

$$f(x) = \sum_{n=0}^{\infty} a_n(\lambda) R_n^{(\alpha,\beta)}(x/\lambda), \tag{1.1}$$
$$\epsilon \leqq x/\lambda \leqq 1 - \epsilon, \quad \epsilon > 0; \quad \alpha > -1, \quad \beta > -1,$$

where $R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x - 1)$ and the latter is the usual notation for the Jacobi polynomial [1, Ch. 10]. Various techniques are available for the determination of the coefficients $a_n(\lambda)$. In this connection, see, for example, the references [2, 3, 4, 5, 6, 7].

Suppose that $f(x)$ satisfies the above conditions for $1 \leqq x/\lambda \leqq \infty$ where $|\arg \lambda| < \varphi$. Then we may write

$$f(x) = \sum_{n=0}^{\infty} b_n(\lambda) R_n^{(\alpha,\beta)}(\lambda/x), \tag{1.2}$$
$$\epsilon \leqq \lambda/x \leqq 1 - \epsilon, \quad \epsilon > 0; \quad \alpha > -1, \quad \beta > -1.$$

If $f(x)$ has an asymptotic expansion of the form

$$f(x) \sim \sum_{n=0}^{\infty} c_n x^{-n}, \qquad x \to \infty, \qquad |\arg x| < \varphi, \tag{1.3}$$

then (1.2) may be interpreted as a summability process which converts the generally divergent expansion (1.3) into a convergent expansion. If $f(x)$ in (1.3) is of hypergeometric type,* then the coefficients $b_n(\lambda)$ may be found formally at least using the procedures [5, 6]. These yield for $b_n(\lambda)$ an asymptotic series in λ which is also of hypergeometric type. The asymptotic representation for $b_n(\lambda)$ in general is not suitable for computation. We are confronted with two problems: one is the interpretation of the asymptotic series for $b_n(\lambda)$, and the other is the computation of $b_n(\lambda)$.

In this paper, we show how both problems can be solved for a confluent hypergeometric function. Actually we derive a representation for $b_n(\lambda)$ when $f(x)$ is the G-function, which includes the confluent hypergeometric function as a special case. Our computational scheme for $b_n(\lambda)$ is exhibited only when $f(x)$ is a confluent hypergeometric function, although the ideas involved can be extended to cover other special cases of the G-function as well.

In Section II, we prove an expansion theorem of the form (1.2) when $f(x)$ is the

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* For the definition and properties of generalized hypergeometric series including the G-function as well as other notations used in this paper, see [1, Chs. 4, 5, 6].

G-function and show how both convergent and asymptotic representations for $b_n(\lambda)$ may be derived. These results are specialized in Section III for the case when $f(x)$ is a confluent hypergeometric function, and in Section IV it is shown how $b_n(\lambda)$ may be computed by a recursion scheme. Finally, in Section V, we tabulate coefficients for the cases where $R_n^{(\alpha,\beta)}(x)$ is the shifted Chebyshev polynomial and $f(x)$ is the error function, the exponential, sine and cosine integrals, and the Bessel functions $K_0(x)$ and $K_1(x)$.

2. Expansion of the G-Function. The G-function is given by the Mellin-Barnes integral

$$(2.1)\qquad G_{p,q}^{m,k}\left(\lambda x \,\middle|\, {}^{a_p}_{b_q}\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{k}\Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + s)\prod_{j=k+1}^{p}\Gamma(a_j - s)}(\lambda x)^s\,ds,$$

where an empty product is interpreted as 1, $0 \leqq m \leqq q$, $0 \leqq n \leqq p$ and the parameters are such that no pole of $\Gamma(b_j - s)$, $j = 1, 2, \cdots, m$ coincides with any pole of $\Gamma(1 - a_h + s)$, $h = 1, 2, \cdots, k$. We assume x is real and the path L runs parallel to the imaginary axis and is indented so that the poles of $\Gamma(b_j - s)$, $j = 1, 2, \cdots, m$, are to the right, and all the poles of $\Gamma(1 - a_h + s)$, $h = 1, 2, \cdots, k$, to the left of L. The integral converges if $p + q < 2(m + k)$ and $|\arg\lambda| < (m + k - p/2 - q/2)\pi$. For a treatment of the G-function, see [1, Ch. 5].

Now from [1, 10.20(3)] we have the expansion

$$(2.2)\qquad \begin{aligned} x^s &= \Gamma(\beta - s + 1)\Gamma(1 - s) \\ &\times \sum_{n=0}^{\infty}\frac{(2n + \alpha + \beta + 1)(n + \beta + 1)_\alpha}{\Gamma(n + \alpha + \beta + 2 - s)\Gamma(1 - s - n)} R_n^{(\alpha,\beta)}(1/x), \quad 1 < x < \infty, \end{aligned}$$

uniformly for $\mathrm{Re}\,(s) \leqq \mu - \delta$, $\delta > 0$, $\mu = \min\,(\beta + 1, \beta/2 + \frac{3}{4})$, $\alpha > -1$, $\beta > -1$. Put (2.2) in (2.1) and integrate along the path from $\mu - \delta - i\infty$ to $\mu - \delta + i\infty$. We then get

THEOREM I. *Let*

1. α, β *and* x *be real*, $\alpha > -1$, $\beta > -1$, $1 < x < \infty$.

Let a real positive δ *exist such that*

2. (a) $\mathrm{Re}\,(a_j - 1) < \mu - \delta$, $j = 1, 2, \cdots k$; (b) $\mathrm{Re}\,(b_j) > \mu - \delta$, $j = 1, 2, \cdots m$, $\mu - \delta < 1$, $\mu = \min\,(\beta + 1, \beta/2 + \frac{3}{4})$.

3. $p + q < 2(m + k)$, $|\arg\lambda| < (m + k - p/2 - q/2)\pi$, $\lambda \neq 0$, $0 \leqq m \leqq q$, $0 \leqq k \leqq p$.

Then

$$(2.3)\qquad \begin{aligned} G_{p,q}^{m,k}\left(\lambda x_b^a \,\middle|\, {}^{p}_{q}\right) &= \sum_{n=0}^{\infty}(2n + \alpha + \beta + 1)(n + \beta + 1)_\alpha \\ &\times G_{p+2,q+2}^{m+2,k}\left(\lambda \,\middle|\, {}^{a_p, 1-n, n+\alpha+\beta+2}_{1, \beta+1, b_q}\right) R_n^{(\alpha,\beta)}(1/x). \end{aligned}$$

Remark. Assumptions 2 above insure the separation of poles and specify the regions in which they lie according to the remarks surrounding (2.1). Notice, however, that poles of $\Gamma(b_j - s)$ may lie to the left of the contour. They may be excluded by indentations since they lie in a region where the series for x^s converges uniformly, provided they do not coincide with any of the poles of $\Gamma(1 - a_h + s)$. Hence, we may replace 2(b) by the weaker but more complicated condition

2(b)* $1 + \delta_{j-2} - a_h \neq 0, -1, -2, \cdots,$

$$j = 1, 2, \cdots m + 2, h = 1, 2, \cdots k, \delta_{-2} = 1, \delta_{-1} = \beta + 1, \delta_{j-2} = b_j, j > 1.$$

Notice from the definition of the G-function that

$$G_{p+2,q+2}^{m+2,k}\left(\lambda \,\middle|\, {}^{a_p, 1-n, n+\alpha+\beta+2}_{1, \beta+1, b_q}\right) = (-)^n G_{p+2,q+2}^{m+1,k+1}\left(\lambda \,\middle|\, {}^{1-n, a_p, n+\alpha+\beta+2}_{\beta+1, b_q, 1}\right). \tag{2.4}$$

If $|\arg \lambda| < \frac{1}{2}(p - q + 1)\pi$, an asymptotic representation for the coefficients of $R_n^{(\alpha,\beta)}(1/x)$ in (2.3) follows by application of a result in [1, 5.3(6)]. An ascending series representation follows when [1, 5.3(5)] is applied to the right-hand side of (2.4).

3. Expansion of a Confluent Hypergeometric Function. We consider the function [1, Ch. 6],

$$(\lambda x)^a \psi(a, c \mid \lambda x) = \{\Gamma(a)\Gamma(\sigma)\}^{-1} G_{1,2}^{2,1}\left(\lambda x \,\middle|\, {}^{1}_{a,\sigma}\right), \quad \sigma = a + 1 - c. \tag{3.1}$$

Also, denote by $T_n^*(x)$ the shifted Chebyshev polynomial

$$T_n^*(x) = T_n(2x - 1) = \frac{n!}{(\frac{1}{2})_n} R_n^{(-1/2,-1/2)}(x). \tag{3.2}$$

From Theorem I, we get

THEOREM II. *Let*

1. $1 \leqq x \leqq \infty$;
2. $\sigma \neq 0, -1, -2, \cdots; a \neq 0, -1, -2, \cdots;$
3. $|\arg \lambda| < 3\pi/2, \lambda \neq 0.$

Then

$$(\lambda x)^a \psi(a, c \mid \lambda x) = \sum_{n=0}^{\infty} C_n(\lambda) T_n^*(1/x), \tag{3.3}$$

where

$$C_n(\lambda) = \frac{\epsilon_n}{\pi^{1/2}\Gamma(a)\Gamma(\sigma)} G_{3,4}^{4,1}\left(\lambda \,\middle|\, {}^{1, 1-n, n+1}_{1, 1/2, a, \sigma}\right), \quad \epsilon_0 = 1, \epsilon_n = 2, n > 0, \tag{3.4}$$

or

$$C_n(\lambda) = \frac{\epsilon_n(-)^n}{\pi^{1/2}\Gamma(a)\Gamma(\sigma)} G_{2,3}^{3,1}\left(\lambda \,\middle|\, {}^{1-n, n+1}_{1/2, a, \sigma}\right). \tag{3.5}$$

Also, if none of the quantities $\frac{1}{2}$, a and σ differ by an integer

$$\begin{aligned} C_n(\lambda) = \frac{\epsilon_n(-)^n}{\pi^{1/2}} \Bigg\{ & (a)_{-1/2}(\sigma)_{-1/2}\lambda^{1/2} {}_2F_2\left({}^{n+1/2, -n+1/2}_{3/2-a, 3/2-\sigma} \middle| \lambda\right) \\ & + \frac{\Gamma(\frac{1}{2} - a)(a)_n(\sigma)_{-a}}{\Gamma(n + 1 - a)} \lambda^a {}_2F_2\left({}^{n+a, -n+a}_{a+1/2, a-\sigma+1} \middle| \lambda\right) \\ & + \frac{\Gamma(\frac{1}{2} - \sigma)(\sigma)_n(a)_{-\sigma}}{\Gamma(n - \sigma + 1)} \lambda^\sigma {}_2F_2\left({}^{n+\sigma, -n+\sigma}_{\sigma+1/2, \sigma-a+1} \middle| \lambda\right) \Bigg\}, \end{aligned} \tag{3.6}$$

and

$$(3.7)\quad C_n(\lambda) \sim \frac{\epsilon_n(-)^n(a)_n(\sigma)_n}{n!(4\lambda)^n}\, {}_3F_1\left(\begin{matrix} n+1/2, n+a, n+\sigma \\ 2n+1 \end{matrix} \middle| -\frac{1}{\lambda}\right), \quad |\lambda| \to \infty, \quad |\arg \lambda| < \pi.$$

Remark. Condition 1 of Theorem I is conservative. By an appeal to the convergence properties of expansions in Chebyshev polynomials [7], the range of x may be extended to give condition 1 above.

Since (3.3) converges,

$$(3.8)\qquad \lim_{n\to\infty} C_n(\lambda) = 0.$$

For later use, we record the fact that

$$(3.9)\qquad \lim_{x\to\infty} (\lambda x)^a \psi(a, c \mid \lambda x) = 1, \qquad |\arg \lambda| < 3\pi/2.$$

4. Calculation of the Coefficients $C_n(\lambda)$. Let

$$(4.1)\qquad \varphi_{1,n}(\lambda) = \frac{(-)^n}{\epsilon_n} C_n(\lambda).$$

Following the method developed in [8], we can show from the representation (3.7) that $\varphi_{1,n}(\lambda)$ satisfies the recursion relation

$$(4.2)\quad \varphi_n(\lambda) + (A_n + B_n\lambda)\varphi_{n+1}(\lambda) + (C_n + D_n\lambda)\varphi_{n+2}(\lambda) + E_n\varphi_{n+3}(\lambda) = 0,$$

where

$$(4.3)\quad \begin{aligned} A_n &= (2n+2)\left[1 - \frac{(n+\frac{3}{2})(n+a+1)(n+\sigma+1)}{(n+2)(n+a)(n+\sigma)}\right], \\ B_n &= D_n = -4(n+1)/(n+a)(n+\sigma), \\ C_n &= -1 + [2(n+1)(2n+3)/(n+a)(n+\sigma)], \\ E_n &= -(n+1)(n-a+3)(n-\sigma+3)/(n+2)(n+a)(n+\sigma). \end{aligned}$$

We prove that the coefficients may be readily evaluated using (4.2) in the backward direction. This backward recursion technique has been treated by many authors [9], [10], [11], [12], [13]. The idea is as follows.

For fixed λ, arbitrary η and ν sufficiently large set

$$(4.4)\qquad \varphi_\nu^{(\nu)}(\lambda) = \varphi_{\nu-1}^{(\nu)}(\lambda) = 0,$$

$$(4.5)\qquad \varphi_{\nu-2}^{(\nu)}(\lambda) = \eta.$$

The sequence $\varphi_{\nu-3}^{(\nu)}(\lambda), \cdots, \varphi_n^{(\nu)}(\lambda), \cdots \varphi_1^{(\nu)}(\lambda), \varphi_0^{(\nu)}(\lambda)$ is generated from (4.2). Using (3.9) and

$$(4.6)\qquad T_n^*(0) = (-)^n$$

in (3.3) we would expect that if

$$(4.7)\qquad \omega_\nu = \sum_{n=0}^{\nu-2} \epsilon_n\varphi_n^{(\nu)}(\lambda),$$

then

$$C_n(\lambda) \sim (-)^n \epsilon_n \varphi_n^{(\nu)}(\lambda)/\omega_\nu , \tag{4.8}$$

with increasing accuracy as $\nu \to \infty$. In fact if we define

$$\varphi_{1,n}^{(\nu)}(\lambda) = \varphi_{1,0}(\lambda)\varphi_n^{(\nu)}(\lambda)/\varphi_0^{(\nu)}(\lambda), \tag{4.9}$$

we have:

THEOREM III. *Let* $|\arg \lambda| < \pi$, $\lambda \neq 0$, *and neither* a *nor* σ *be a negative integer or zero. Then*

$$\lim_{\nu\to\infty} \varphi_{1,n}^{(\nu)}(\lambda) = \varphi_{1,n}(\lambda). \tag{4.10}$$

Proof. Denote by $\varphi_{1,n}(\lambda)$, $\varphi_{2,n}(\lambda)$ and $\varphi_{3,n}(\lambda)$ the three linearly independent solutions of (4.2); $\varphi_{1,n}(\lambda)$ is the solution we wish to calculate. We may write*

$$\varphi_n^{(\nu)} = \xi_1^{(\nu)}\varphi_{1,n} + \xi_2^{(\nu)}\varphi_{2,n} + \xi_3^{(\nu)}\varphi_{3,n} , \qquad n < \nu - 2, \tag{4.11}$$

and the conditions (4.4) and (4.5) give

$$0 = \xi_1^{(\nu)}\varphi_{1,\nu} + \xi_2^{(\nu)}\varphi_{2,\nu} + \xi_3^{(\nu)}\varphi_{3,\nu} , \tag{4.12}$$

$$0 = \xi_1^{(\nu)}\varphi_{1,\nu-1} + \xi_2^{(\nu)}\varphi_{2,\nu-1} + \xi_3^{(\nu)}\varphi_{3,\nu-1} , \tag{4.13}$$

$$\eta = \xi_1^{(\nu)}\varphi_{1,\nu-2} + \xi_2^{(\nu)}\varphi_{2,\nu-2} + \xi_3^{(\nu)}\varphi_{3,\nu-2} , \tag{4.14}$$

where $\xi_1^{(\nu)}$, $\xi_2^{(\nu)}$ and $\xi_3^{(\nu)}$ are independent of n.

$$\xi_2^{(\nu)}/\xi_1^{(\nu)} = \gamma_\nu , \qquad \xi_3^{(\nu)}/\xi_1^{(\nu)} = \delta_\nu , \tag{4.15}$$

$$\gamma_\nu = [-\varphi_{1,\nu}\varphi_{3,\nu-1} + \varphi_{1,\nu-1}\varphi_{3,\nu}]/\tau_\nu , \tag{4.16}$$

$$\delta_\nu = [-\varphi_{2,\nu}\varphi_{1,\nu-1} + \varphi_{1,\nu}\varphi_{2,\nu-1}]/\tau_\nu , \tag{4.17}$$

$$\tau_\nu = [\varphi_{2,\nu}\varphi_{3,\nu-1} - \varphi_{3,\nu}\varphi_{2,\nu-1}]. \tag{4.18}$$

Thus

$$\varphi_{1,n}^{(\nu)} = \frac{\varphi_{1,n}\{1 + (\gamma_\nu\varphi_{2,n}/\varphi_{1,n}) + (\delta_\nu\varphi_{3,n}/\varphi_{1,n})\}}{\{1 + (\gamma_\nu\varphi_{2,n}/\varphi_{1,0}) + (\delta_\nu\varphi_{3,0}/\varphi_{1,0})\}}. \tag{4.19}$$

We will show that

$$\lim_{\nu\to\infty} \gamma_\nu = \lim_{\nu\to\infty} \delta_\nu = 0. \tag{4.20}$$

Equation (3.8) gives

$$\lim_{\nu\to\infty} \varphi_{1,\nu} = 0. \tag{4.21}$$

It may be directly verified that

$$\varphi_{2,n} = {}_2F_2\left({n+1/2, -n+1/2 \atop 3/2-a, 3/2-\sigma} \,\middle|\, \lambda\right), \tag{4.22}$$

is also a solution of (4.2). From [14] we have

$$\varphi_{2,n} = C_1 n^{2/3[a+\sigma-2]} \exp\left[\tfrac{3}{2} n^{2/3}\lambda^{1/3}\right]\left[1 + O\left(\frac{1}{n}\right)\right], \qquad |\arg \lambda| < \pi, \tag{4.23}$$

* Henceforth we write, $\xi_1^{(\nu)}(\lambda) = \xi_1^{(\nu)}$, $\varphi_{1,n}(\lambda) = \varphi_{1,n}$, etc.

TABLE I
Coefficients for the Series

$$-Ei(-x) = \int_x^\infty \frac{e^{-t}}{t}\,dt = \frac{e^{-x}}{x}\sum_{n=0}^{\infty} A_n T_n^*\left(\frac{4}{x}\right), \qquad 4 \leqq x \leqq \infty,$$

$$Erfc(x) = \int_x^\infty e^{-t^2}\,dt = \frac{e^{-x^2}}{2x}\sum_{n=0}^{\infty} B_n T_{2n}\left(\frac{2}{x}\right), \qquad 2 \leqq x \leqq \infty.$$

n	A_n	B_n	n	A_n	B_n
0	0.90535 40999 62349 15873 (00)	0.94960 80415 75614 24493 (00)	19	−0.47291 68 (−13)	−0.83782 0 (−14)
1	−0.86481 17855 25987 1490 (−01)	−0.47027 51541 55887 6766 (−01)	20	0.14637 62 (−13)	0.25450 2 (−14)
2	0.72241 01543 74659 475 (−02)	0.30329 41457 65811 336 (−02)	21	−0.46173 9 (−14)	−0.78866 (−15)
3	−0.80975 59457 55738 62 (−03)	−0.29024 22664 89234 09 (−03)	22	0.14827 1 (−14)	0.24900 (−15)
4	0.10999 13443 26613 89 (−03)	0.35220 95068 75272 1 (−04)	23	−0.48417 (−15)	−0.8001 (−16)
5	−0.17173 32998 93776 7 (−04)	−0.50403 49016 87590 (−05)	24	0.16062 (−15)	0.2614 (−16)
6	0.29856 27514 47928 (−05)	0.81642 08349 9637 (−06)	25	−0.5409 (−16)	−0.867 (−17)
7	−0.56596 49145 7719 (−06)	−0.14583 31253 5019 (−06)	26	0.1847 (−16)	0.292 (−17)
8	0.11526 80839 7141 (−06)	0.28219 97993 581 (−07)	27	−0.640 (−17)	−0.100 (−17)
9	−0.24950 30440 269 (−07)	−0.58403 61925 03 (−08)	28	0.224 (−17)	0.35 (−18)
10	0.56923 24201 83 (−08)	0.12803 21362 01 (−08)	29	−0.80 (−18)	−0.12 (−18)
11	−0.13599 57664 81 (−08)	−0.29508 80097 7 (−09)	30	0.29 (−18)	0.4 (−19)
12	0.33846 62888 8 (−09)	0.71082 35787 (−10)	31	−0.10 (−18)	−0.2 (−19)
13	−0.87378 53904 (−10)	−0.17809 90369 (−10)	32	0.4 (−19)	0.1 (−19)
14	0.23315 88663 (−10)	0.46230 4869 (−11)	33	−0.1 (−19)	
15	−0.64114 8105 (−11)	−0.12391 4208 (−11)			
16	0.18122 4698 (−11)	0.34199 361 (−12)			
17	−0.52538 318 (−12)	−0.96955 55 (−13)			
18	0.15592 183 (−12)	0.28176 05 (−13)			

TABLE II

Coefficients for the Series

$$K_0(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x} \sum_{n=0}^{\infty} A_n T_n^*\left(\frac{2}{x}\right), \qquad 2 \leqq x \leqq \infty,$$

$$K_1(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x} \sum_{n=0}^{\infty} B_n T_n^*\left(\frac{2}{x}\right), \qquad 2 \leqq x \leqq \infty.$$

n	A_n		B_n		n	A_n		B_n	
0	0.97354 00764 30036 78069	(+00)	1.08537 28165 51726 12040	(+00)	16	0.64093 20	(−13)	−0.71623 74	(−13)
1	−0.25091 95450 33808 0930	(−01)	0.82919 14491 55864 9624	(−01)	17	−0.17772 98	(−13)	0.19764 83	(−13)
2	0.12525 86114 67721 930	(−02)	−0.22802 07949 89514 525	(−02)	18	0.50586 5	(−14)	−0.56010 2	(−14)
3	−0.10252 45722 44517 42	(−03)	0.15575 94482 17418 05	(−03)	19	−0.14749 6	(−14)	0.16266 5	(−14)
4	0.11130 34099 23675 6	(−04)	−0.15448 62470 24490 3	(−04)	20	0.43980	(−15)	−0.48328	(−15)
5	−0.14615 29450 74297	(−05)	0.19200 97185 68380	(−05)	21	−0.13390	(−15)	0.14666	(−15)
6	0.22075 97885 5320	(−06)	−0.27941 60297 7437	(−06)	22	0.4157	(−16)	−0.4540	(−16)
7	−0.37185 32935 143	(−07)	0.45807 22385 967	(−07)	23	−0.1315	(−16)	0.1431	(−16)
8	0.68410 89366 29	(−08)	−0.82547 24141 10	(−08)	24	0.423	(−17)	−0.459	(−17)
9	−0.13544 36576 47	(−08)	0.16077 77088 92	(−08)	25	−0.138	(−17)	0.150	(−17)
10	0.28543 50058 7	(−09)	−0.33434 19366 8	(−09)	26	0.46	(−18)	−0.50	(−18)
11	−0.63491 57811	(−10)	0.73551 51365	(−10)	27	−0.15	(−18)	0.17	(−18)
12	0.14808 33144	(−10)	−0.16994 68453	(−10)	28	0.5	(−19)	−0.6	(−19)
13	−0.36021 2795	(−11)	0.41008 3914	(−11)	29	−0.2	(−19)	0.2	(−19)
14	0.90986 015	(−12)	−0.10286 1200	(−11)	30	0.1	(−19)	−0.1	(−19)
15	−0.23777 733	(−12)	0.26716 524	(−12)					

TABLE III
Coefficients for the Series

$$Si(x) = \int_x^\infty \frac{\sin t}{t}\,dt = \sum_{n=0}^{\infty} \left\{ \frac{A_n \cos x}{x} + \frac{B_n \sin x}{x} \right\} T_n^*\left(\frac{4}{x}\right), \qquad 4 \leqq x \leqq \infty,$$

$$Ci(x) = \int_x^\infty \frac{\cos t}{t}\,dt = \sum_{n=0}^{\infty} \left\{ \frac{B_n \cos x}{x} - \frac{A_n \sin x}{x} \right\} T_n^*\left(\frac{4}{x}\right), \qquad 4 \leqq x \leqq \infty.$$

n	A_n	B_n	n	A_n	B_n
0	0.96578 82803 51851 83021 (00)	0.10728 86713 38433 09526 (00)	21	0.17469 921 (−12)	−0.81934 01 (−13)
1	−0.43060 83777 85967 3425 (−01)	0.99693 56055 36349 5732 (−01)	22	−0.38470 01 (−13)	0.60587 82 (−13)
2	−0.73143 71174 81046 083 (−02)	−0.81628 39500 94241 970 (−02)	23	0.20193 (−15)	−0.27093 29 (−13)
3	0.14705 23578 98680 654 (−02)	−0.29696 08630 56773 29 (−03)	24	0.53597 7 (−14)	0.88808 3 (−14)
4	−0.98657 68573 27002 1 (−04)	0.22891 94548 45829 18 (−03)	25	−0.35606 0 (−14)	−0.18751 4 (−14)
5	−0.22743 20220 46550 8 (−04)	−0.41721 77635 53092 6 (−04)	26	0.15788 2 (−14)	−0.8110 (−16)
6	0.98240 25732 25254 (−05)	0.21254 28930 87307 (−05)	27	−0.52547 (−15)	0.34480 (−15)
7	−0.18973 43014 87133 (−05)	0.13157 50436 91368 (−05)	28	0.11378 (−15)	−0.22564 (−15)
8	0.10063 43594 1558 (−06)	−0.55848 57495 6974 (−06)	29	0.512 (−17)	0.10258 (−15)
9	0.80819 36482 224 (−07)	0.12353 72625 6029 (−06)	30	−0.2244 (−16)	−0.3571 (−16)
10	−0.38976 28287 529 (−07)	−0.10318 72179 187 (−07)	31	0.1532 (−16)	0.845 (−17)
11	0.10335 65032 550 (−07)	−0.50159 03675 67 (−08)	32	−0.732 (−17)	−0.3 (−19)
12	−0.14104 34487 59 (−08)	0.30915 99889 01 (−08)	33	0.273 (−17)	−0.146 (−17)
13	−0.25232 07840 0 (−09)	−0.10080 57370 10 (−08)	34	−0.73 (−18)	0.110 (−17)
14	0.25699 83132 6 (−09)	0.20289 59643 1 (−09)	35	0.6 (−19)	−0.56 (−18)
15	−0.10597 88925 4 (−09)	−0.27237 6669 (−11)	36	0.9 (−19)	0.23 (−18)
16	0.28970 03157 (−10)	−0.19967 52281 (−10)	37	−0.8 (−19)	−0.7 (−19)
17	−0.41023 1426 (−11)	0.11219 38506 (−10)	38	0.5 (−19)	0.1 (−19)
18	−0.10437 6937 (−11)	−0.40081 1186 (−11)	39	−0.2 (−19)	0.0 (−20)
19	0.10994 1845 (−11)	0.96702 841 (−12)	40	0.1 (−19)	−0.1 (−19)
20	−0.52214 239 (−12)	−0.71861 28 (−13)			

where C_1 is independent of n. The third linearly independent solution of (4.2) is the $L_{2,2}(-\lambda)$ term appearing in [15, 1.3.3(15)] which arises in the asymptotic expansion of (4.22) for large λ. A limit process, explained in [15, 1.3.4] is used to obtain $\varphi_{3,n}$, but our discussion here is necessarily brief. We need only the estimate

$$(4.24) \qquad \varphi_{3,n} = \frac{C_2\Gamma(n+a-1)\Gamma(n+\sigma-1)}{(4\lambda)^n n!}\left[1+O\left(\frac{1}{n^2}\right)\right],$$

where C_2 is independent of n. Thus

$$(4.25) \qquad \lim_{\nu\to\infty} |\varphi_{2,\nu}| = \lim_{\nu\to\infty} |\varphi_{3,\nu}| = \infty.$$

Also, from (4.23) and (4.24), we have

$$(4.26) \qquad \tau_\nu = -\varphi_{2,\nu}\varphi_{3,\nu}\left[1+O\left(\frac{1}{\nu}\right)\right].$$

Hence (4.20) is easily shown and the statement (4.10) follows from (4.19).

5. Tables. Tables I–III contain coefficients to 20 D for the expansions of several important cases of the confluent hypergeometric function [1, 6.9]. Coefficients corresponding to different ranges of the independent variable as well as those for other functions, e.g., $J_\nu(x)$ and $Y_\nu(x)$, are under construction and the present tables are selected examples only. The expansions are readily evaluated using a nesting procedure described in [4], [7]. For similar expansions, see [7], and for many Chebyshev expansions of functions over a finite interval, see [2]–[6] and the references given there. The number in parenthesis after each entry in the tables is the power of 10 by which the entry is to be multiplied.

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