A Finite Difference Exponential Approximation Method

By J. W. Layman

1. Introduction. Numerous approximating or interpolating methods are used in numerical analysis, among these being the polynomial, rational function, trigonometric, and exponential function methods. (For a directory of methods see [1, pp. 502–505].) The polynomial formulas are the most frequently used and simple finite-difference methods are available for their application. It might be useful from a practical point of view and also interesting from a pure finite-difference standpoint to have available a similar method with regard to a certain exponential approximating function.

Following the introduction of new difference operations in sections 2–4 below, we describe in section 5 a finite-difference method whereby the coefficients a_n can be determined such that the exponential polynomial

$$P(k) = a_0 0^k + a_1 1^k + a_2 2^k + \cdots + a_N N^k = \sum_{n=0}^{N} a_n n^k \tag{1}$$

takes on $N + 1$ prescribed values for $k = 0, 1, \cdots, N$.

An illustrative numerical example is presented in section 6 and possible generalization pointed out in section 7.

2. Diagonal Differences. We make use of the shift operator E^n defined by the equation

$$E^n f(k) = f(k + n)$$

and the difference operator Δ defined by the equation

$$\Delta f(k) = f(k + 1) - f(k) = (E - 1)f(k), \tag{2}$$

with higher order differences defined in the usual iterative manner.

We now define a new difference operation as follows:

The *diagonal difference* $S_{(k)}f(t)$ of a function $f(k)$ defined for discrete integral values $k = 0, 1, 2, \cdots, N$, is the function whose value at k is the k^{th} difference of $f(t)$ at $t = 0$. In symbols

$$S_{(k)}f(t) = \Delta^k f(0). \tag{3}$$

When there is no risk of ambiguity, we may write instead

$$Sf(k) = \Delta^k f(0). \tag{3'}$$

Then by $Sf(k + a)$ we shall mean

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$$S_{(k)}f(t + a) = \Delta^k f(a).$$

In the next section we show that $Sf(k + a)$ is not in general the same as $E^a Sf(k)$.

The higher order diagonal differences may be defined by iteration:

$$S^n_{(k)}f(t) = S_{(k)}S^{n-1}_{(t)}f(t').$$

Again, when there is no danger of ambiguity we write simply

$$S^n f(k) = SS^{n-1}f(k).$$

A word of caution in connection with the higher order diagonal differences might be in order. It must be remembered that the exponent on Δ is an argument with respect to operators containing E. This is apparent from the definition (3). As an example, consider

$$S^2 f(2) = S_{(2)}S_{(t)}f(t').$$

The right hand side may be written in either of the two forms

$$\Delta^2 S_{(0)}f(t'), \qquad S_{(2)}\Delta^t f(0),$$

both of which reduce to $\Delta^2\Delta^0 f(0)$, where the exponent in Δ^0 is an argument with respect to the operator Δ^2 to its left. Hence,

$$\Delta^2\Delta^0 f(0) = (E - 1)^2\Delta^0 f(0) = \Delta^2 f(0) - 2\Delta f(0) + f(0).$$

To conclude this section we illustrate the tabular calculation of $S(3^k)$ and $S^2(3^k)$ for $k = 0, 1, 2, 3$. The tabulation is as follows:

k	3^k				$S(3^k) = 2^k$				$S^2(3^k) = 1^k$
0	$\underline{1}$				$\underline{1}$				1
1	3	$\underline{2}$			2	$\underline{1}$			1
2	9	6	$\underline{4}$		4	2	$\underline{1}$		1
3	27	18	12	$\underline{8}$	8	4	2	$\underline{1}$	1

Thus it is seen that in tabular form the operation of taking a diagonal difference is especially simple, involving merely the determination of the diagonal of so-called leading differences.

3. Some Properties of Diagonal Differences. In order to investigate some of the properties of diagonal differences, it is convenient to rewrite formula (3) as follows:

$$Sf(k) = \Delta^k f(0) = (E - 1)^k f(0). \tag{4}$$

The second diagonal difference can then be written

$$\begin{aligned} S^2 f(k) &= S_{(k)}S_{(t)}f(t') = \Delta^k\Delta^0 f(0) \\ &= (E - 1)^k\Delta^0 f(0) = (\Delta - 1)^k f(0) \\ &= (E - 2)^k f(0). \end{aligned}$$

By induction on n it can be shown that in general

$$S^n f(k) = (E - n)^k f(0). \tag{5}$$

The diagonal differences of $f(k) = a^k$ will be important for our exponential approximation in section 5. They can be obtained very easily as follows:

$$S^n a^k = (E - n)^k a^0 = (a - n)^k. \tag{6}$$

As stated earlier $Sf(k + a) = SE^a f(k)$ is not in general $E^a Sf(k)$. That this is the case is easily shown in the following manner:

$$SEf(k) = \Delta^k Ef(0) = \Delta^k(\Delta + 1)f(0) = (\Delta^{k+1} + \Delta^k)f(0) = (E + 1)Sf(k).$$

Many other properties and formulas relating to diagonal differences may be developed. Some of these are

$$S^n[af(k) + bg(k)] = aS^n f(k) + bS^n g(k)$$
$$S^n[k^{(r)}f(k)] = k^{(r)}E^{-r}S^n f(k + r)$$
$$S^m[m^k f(k)] = m^k S^1 f(k).$$

The expression 0^k will occur in the following and for consistency must be defined to be unity when $k = 0$ and zero when $k > 0$. This is suggested in the following manner. If formula (6) is required to hold for $a = 1$ and $n = 1$, then

$$0^k = S(1^k) = (E - 1)^k 1^0.$$

Setting $k = 0$ gives $0^0 = (E - 1)^0 1^0 = 1$.

4. The Operator T^n. Without some modification, full advantage cannot be taken of the result in formula (6), which states that the base of an exponential function decreases by unity as a result of taking the diagonal difference. This is because the process of taking higher and higher diagonal differences of a^k does not terminate as does the process of taking higher and higher differences of $k^{(n)}$. This can be seen very easily:

$$S(a^k) = (a - 1)^k, \cdots, S^a(a^k) = 0^k, \cdots, S^{a+r}(a^k) = (-r)^k, \quad \text{etc.}$$

What is needed is some procedure for discarding the entry at which the zero-base exponential takes on a non-zero value, thus preventing the propagation of non-zero entries in higher diagonal differences. For the function $f(k) = 3^k$, this can be accomplished by disregarding the $k = 0$ entry in $S^3(3^k)$. The resulting diagonal difference is then the constant function zero.

k	$S^3(3^k)$		
0	$\underline{1}$		
1	0		
2	0	0	
3	0	0	0

Rather than thinking in terms of disregarding certain entries, it is more convenient to think of the process as equivalent to first shifting the column of entries upward, then taking the diagonal difference. This procedure applied to 0^k gives

$$SE(0^k) = S(0^{k+1}) = (0)S(0^k) = 0(-1^k) = 0,$$

a result which is agreeably zero for all k, whereas $S(0^k) = (-1)^k$.

Now define a new operation T^n as follows:

$$T^n f(k) = (SE)^n f(k). \tag{7}$$

It can easily be shown that

$$T^n(a^k) = a^{(n)}(a-n)^k. \tag{8}$$

In particular

$$T^a(a^k) = a!0^k \tag{9}$$

and

$$T^{a+1}(a^k) = 0 \tag{10}$$

when a is an integer.

5. Exponential Approximation. We are now in a position where we are able to determine the coefficients a_i such that the function.

$$f(k) = a_0 0^k + a_1 1^k + a_2 2^k + \cdots + a_N N^k \tag{11}$$

takes on $N+1$ given values for $N+1$ values of the argument spaced at unit intervals and translated to the origin.

Using formula (8), we have

$$\begin{aligned}
T^0 f(k) &= a_0 0^k + a_1 1^k + a_2 2^k + a_3 3^k + \cdots + a_N N^k \\
T^1 f(k) &= 0 + a_1 0^k + 2a_2 1^k + 3a_3 2^k + \cdots + N a_N (N-1)^k \\
T^2 f(k) &= 0 + 2a_2 0^k + 6a_3 1^k + \cdots + N(N-1)a_n(N-2)^k \\
T^3 f(k) &= 0 + 6a_3 0^k + \cdots + N(N-1)(N-2)a_N(N-3)^k \\
&\vdots \\
T^N f(k) &= N!\, a_N 0^k.
\end{aligned}$$

Now let

$$g_m = T^m f(0). \tag{12}$$

Then we have:

$$\begin{aligned}
g_o &= a_0 + a_1 + a_2 + a_3 + \cdots + a_n \\
g_1 &= a_1 + 2a_2 + 3a_3 + \cdots + N a_N \\
g_2 &= 2a_2 + 6a_3 + \cdots + N(N-1)a_N \\
g_3 &= 6a_3 + \cdots + N(N-1)(N-2)a_N \\
&\vdots \\
g_N &= N!\, a_N .
\end{aligned}$$

The solution of this system for the a's is not difficult and is

$$a_0 = g_0 - g_1 + \frac{g_2}{2!} - \frac{g_3}{3!} + \cdots + (-1)^N \frac{g_N}{N!}$$

$$a_1 = g_1 - g_2 + \frac{g_3}{2!} - \frac{g_4}{3!} + \cdots + (-1)^{N-1} \frac{g_N}{(N-1)!}$$

$$a_2 = \frac{1}{2!}\left[g_2 - g_3 + \frac{g_4}{2!} - \frac{g_5}{3!} + \cdots + (-1)^{N-2} \frac{g_N}{(N-2)!}\right]$$

and in general

$$a_n = \frac{1}{n!} \sum_{i=0}^{N-n} (-1)^i \frac{g_{i+n}}{i!} . \tag{13}$$

Formulas (11) and (13) provide the desired approximating function.

6. Numerical Example. Let us now use the preceding results on the following data: $f(0) = 2, f(1) = -3, f(2) = 1, f(3) = 4$. A close examination of equations (7) and (12) reveals that in actual practice the values of g_n are obtained as follows from the given data.

k	$f(k)$						
0	$\underline{2} = g_0$						
1	-3			$\underline{-3} = g_1$			
2	1	4		4		$\underline{4} = g_2$	
3	4	3	-1	-1	-5	-5	$\underline{-5} = g_3$

Before taking each set of differences the top number in the column is separated and is not used in taking those differences. This number is one of the g's. The diagonal of differences obtained by using the remaining numbers in this column is then placed in a new column, and the process repeated.

Using equation (9), we get for the a's

$$a_0 = 2 - (-3) + \frac{4}{2!} - \frac{(-5)}{3!} = \frac{47}{6}$$

$$a_1 = -3 - 4 + \frac{-5}{2!} = -\frac{19}{2}$$

$$a_2 = \frac{1}{2!}(4 - (-5)) = \frac{9}{2}$$

$$a_3 = \frac{1}{3!}(-5) = -\frac{5}{6}$$

Therefore the function

$$f(k) = \frac{47}{6} 0^k - \frac{19}{2} 1^k + \frac{9}{2} 2^k - \frac{5}{6} 3^k$$

takes on the given values as can easily be checked.

7. Generalizations. The generalization of the procedures discussed above to arbitrary uniform spacing of the data is straightforward. No investigation has been made for the case of non-uniform spacing. Another generalization, in which the approximating function takes the form

$$f(k) = a_1r_1^k + a_2r_2^k + \cdots + a_Nr_N^k = \sum_{i=1}^{N} a_ir_i^k,$$

where the r_i are completely arbitrary real numbers, can also be made. The a_i can be determined by a modification of the method described in this paper, however the procedure is extremely cumbersome and very decidedly offers no advantage over the obvious Cramer's Rule solution.

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1. F. B. Hildebrand, *Introduction to Numerical Analysis*, McGraw-Hill, New York, 1956.

A New Algorithm for Diagonalizing a Real Symmetric Matrix

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Abstract. The algorithm described in this paper is essentially a Jacobi-like procedure employing Householder and Jacobi orthogonal similarity transformations successively on a real symmetric matrix to obtain, in the limit, a diagonal matrix of eigenvalues. The columns of the product matrix of all the orthogonal transformations, taken in the proper order, form a complete orthonormal set of eigenvectors.

1. Introduction. In this paper we describe a Jacobi-like procedure for diagonalizing a real symmetric matrix by means of orthogonal similarity transformations. The earliest such procedure was proposed by Jacobi in 1847 which involved the use of plane rotations, but required a computer search for the largest off-diagonal element, in absolute value. Later, procedures were proposed in which the off-diagonal elements were annihilated in sequence. The latter method, known as the cyclic Jacobi method, was discussed by Forsythe and Henrici (1). They showed that convergence of this method would only take place if the angle of rotation lay in a closed interval properly contained in the open interval $((-\frac{1}{2})\pi, (+\frac{1}{2})\pi)$. The method described here employs successive Householder and Jacobi orthogonal similarity transformations in a sequential fashion to obtain, as in Jacobi methods, in the limit, a diagonal matrix of eigenvalues. The columns of the product matrix of all the Householder and Jacobi transformations employed form a complete set of orthonormal eigenvectors. Throughout this paper we will confine ourselves to the consideration of real symmetric matrices.

2. General Description of the Algorithm; Definitions and Notations. The general procedure may be described as follows: beginning with an arbitrary sym-

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