4. E. T. GOODWIN, "The evaluation of integrals of the form $\int_{-\infty}^{\infty} e^{-x^2} f(x)\, dx$," *Proc. Cambridge Philos. Soc.*, v. 45, 1949, p. 241.
5. Y. L. LUKE, "Simple formulas for the evaluation of some higher transcendental functions," *J. Math. Phys.*, v. 34, 1956, p. 298.
6. I. A. STEGUN & M. ABRAMOWITZ, "Generation of Bessel functions on high speed computers," *MTAC*, v. 11, 1957, p. 255.
7. A. M. TURING, "A method for the calculation of the zeta-function," *Proc. London Math. Soc.*, Ser. 2, v. 48, 1945, p. 180.
8. G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, 2nd edition, Cambridge University Press, 1944.

Tables of Zeros of Cross Product Bessel Functions $J_p'(\xi)Y_p'(k\xi) - J_p'(k\xi)Y_p'(\xi) = 0$

By Helmut F. Bauer

In the calculation of frequencies for various modes of oscillations in a container comprised by a sector of the annulus between two concentric circular cylinders filled to a height h with incompressible liquid, in wave guide theory and in many other applied problems involving annular or sectional cavities, the roots of the cross product of Bessel functions

$$\Delta_p(\xi) = J_p'(k\xi)Y_p'(\xi) - J_p'(\xi)Y_p'(k\xi) = 0, \qquad p \geqq 0 \tag{1}$$

are of considerable interest. The functions $J_p(\xi)$ and $Y_p(\xi)$ are Bessel functions of order p of the first and second kind respectively. Here k is a parameter and $0 \leqq k < 1$.

If $p = 0$

$$\Delta_0(\xi) = J_1(\xi)Y_1(k\xi) - J_1(k\xi)Y_1(\xi) = 0 \tag{2}$$

and if $k \to 0$, the zeros of $\Delta_p(\xi)$ approach those of $J_p'(\xi) = 0$.

J. McMahon [1] gave an asymptotic expression for the roots of equation (1). However, the lowest root was not known, until D. O. North [2] and H. Buchholz [3] mentioned its existence. Curves showing values of the roots of the expression (1) are shown in a few cases by D. Kirkham [4] for $p = 0, 1, 2, 3, 4$. The purpose of this paper is to extend the range of the index p and present the lowest ten roots for $p = 0(1)25$, $k = 0(0.1)1.0$. Bridge and Angrist [5] give the first eleven zeros for $p = 0(1)12$ and $k = 1.1, 1.2, 1.5, (0.5)5.0$. The present tables extend the range of k. The calculation of the roots was accomplished by interpolation of $\Delta_p(\xi) = 0$.

For the calculation of the derivatives of the Bessel functions the series expanison was used in the argument range $0 \leqq x \leqq 5.0$ for the Bessel function of first kind and $0 \leqq x \leqq 5.75$ for the Bessel function of second kind. For larger arguments the asymptotic expansions were employed. For the cases in which argument and order of Bessel functions are nearly equal the results of Reference [6] have been used. The roots are correct to at least the fourth digit.

Received June 7, 1963.

The present paper presents the zeros for $p = 0(1)11$. Graphs of the zeros versus the parameter k $(0 \leqq k < 1)$ and more extensive tables for $p = 0(1)25$ are available from the author on request.

Geo. C. Marshall Space Flight Center
National Aeronautics & Space Administration
Huntsville, Alabama

J. McMahon, "On the roots of the Bessel and certain related functions," *Ann. of Math.*, v. 9, 1894, p. 23.

2. R. Truell, "Concerning the roots of $J'_n(x)N'_n(kx) - J'_n(kx)N'_n(x) = 0$," *J. Appl. Phys.*, v. 14, 1943, p. 350.

3. H. Buchholz, *Besondere Reihenentwicklungen für eine häufig vorkommende zweireihige Determinante mit Zylinder-Funktionen und ihre Nullstellen*, ZAMM v. 29, 1949, p. 356–367.

4. D. Kirkham, "Graphs and formulas of zeros of cross product Bessel function," *J. Math. Phys.*, v. 36, 1958, p. 371–377.

5. J. F. Bridge & S. W. Angrist, "An extended table of roots of $J'_n(x)Y'_n(\beta x) - J'_n(\beta x)Y'_n(x) = 0$," *Math. Comp.*, v. 16, 1962, p. 192.

6. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, 1952, Section 8.42.

$\Delta_0(\xi)=0$

n \ k	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	3.83171	3.94094	4.23575	4.70577	5.39118	6.39316	7.93008	10.52203	15.73755	31.42923
1	7.01559	7.33057	8.05536	9.10423	10.55773	12.62470	15.74728	20.96939	31.43083	62.83849
2	10.17347	10.74838	11.92658	13.55317	15.76646	18.88893	23.58833	31.43294	47.13383	94.25221
3	13.32369	14.18864	15.82104	18.01998	20.98820	25.16241	31.43576	41.90067	62.83931	125.66702
4	16.47063	17.64331	19.72706	22.49481	26.21548	31.43971	39.28579	52.37010	78.54578	157.08228
5	19.61586	21.10731	23.63949	26.97386	31.44561	37.71895	47.13713	62.84041	94.25276	188.49777
6	22.76008	24.57757	27.55584	31.45540	36.67739	43.99932	54.98923	73.31116	109.96001	219.91338
7	25.90367	28.05219	31.47472	35.93852	41.91022	50.28038	62.84179	83.78220	125.66744	251.32906
8	29.04683	31.52994	35.39534	40.42270	47.14374	56.56194	70.69471	94.25348	141.37499	282.74482
9	32.18968	35.01001	39.31719	44.90764	52.37775	62.84378	78.54780	104.72487	157.08262	314.16056

$\Delta_1(\xi)=0$

n \ k	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	1.84118	1.80347	1.70512	1.58207	1.46178	1.35467	1.26208	1.18237	1.11337	1.05314
1	5.33144	5.13714	4.96086	5.13740	5.65910	6.56494	8.04109	10.59184	15.77771	31.44690
2	8.53632	8.19917	8.43307	9.30827	10.68325	12.70642	15.80106	21.00371	31.45076	62.84733
3	11.70601	11.35880	12.16505	13.68364	15.84808	18.94266	23.62392	31.45574	47.14711	94.25810
4	14.86359	14.63436	15.99323	18.11588	21.04879	25.20249	31.46238	41.91775	62.84927	125.67144
5	18.01553	17.98642	19.86163	22.57071	26.26370	31.47169	39.30706	52.38376	78.55375	157.08582
6	21.16437	21.38369	23.75000	27.03672	31.48568	37.74557	47.15485	62.85178	94.25939	188.50071
7	24.31133	24.80814	27.64965	31.50907	36.71167	44.02211	55.00440	73.32088	109.96569	219.91590
8	27.45705	28.24974	31.55626	35.98536	41.94018	50.30034	62.85507	83.79075	125.67241	251.33127
9	30.60192	31.70269	35.46747	40.46426	47.17035	56.57965	70.70650	94.26106	141.37941	282.74678

$\Delta_2(\xi) = 0$

n \ k	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	3.05424	3.05294	3.03472	2.96850	2.84240	2.68120	2.51595	2.36285	2.22646	2.10622
1	6.70613	6.68669	6.49495	6.27367	6.41600	7.06258	8.36708	10.79885	15.89762	31.50001
2	9.96947	9.88752	9.54946	9.91800	11.05596	12.94941	15.96147	21.10636	31.51055	62.87589
3	13.17037	12.96963	12.89974	14.07466	16.09165	19.10316	23.73041	31.52410	47.18691	94.27578
4	16.34752	16.01187	16.51926	18.40313	21.22993	25.32243	31.54212	41.96895	62.87912	125.68471
5	19.51219	19.10106	20.26958	22.79807	26.40803	31.56748	39.37081	52.42470	78.57762	157.09642
6	22.67158	22.28166	24.08355	27.22504	31.60568	37.82532	47.20795	62.88587	94.27928	188.50955
7	25.82604	25.54689	27.93211	31.66989	36.81439	44.09043	55.04991	73.35016	109.98274	219.92350
8	28.97767	28.87353	31.80144	36.12574	42.02997	50.36011	62.89488	83.81631	125.68733	251.33790
9	32.12733	32.24178	35.68420	40.58883	47.25011	56.63273	70.74189	94.28380	141.39267	282.75268

$\Delta_3(\xi) = 0$

n \ k	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	4.20119	4.20115	4.19906	4.18011	4.10816	3.95776	3.75394	3.53961	3.33900	3.15933
1	8.01524	8.01416	7.96378	7.72126	7.53527	7.84011	8.88932	11.13634	16.09557	31.58836
2	11.34592	11.33836	11.10603	10.92017	11.66607	13.34760	16.22583	21.27642	31.60989	62.91809
3	14.58585	14.55603	14.14931	14.72716	16.49359	19.36843	23.90696	31.63759	47.25324	94.30525
4	17.78875	17.70576	17.43239	18.88109	21.52991	25.52135	31.67463	42.05416	62.92886	125.70680
5	20.97248	20.79509	20.96777	23.17609	26.64750	31.72662	39.47685	52.49288	78.61741	157.11411
6	24.14490	23.84611	24.64817	27.53818	31.80503	37.95793	47.29633	62.94269	94.31243	188.52428
7	27.31006	26.90842	28.40731	31.93739	36.98516	44.20410	55.12567	73.39883	110.01116	219.93612
8	30.47027	30.03007	32.21249	36.35930	42.17934	50.45953	62.96118	83.85893	125.71220	251.34894
9	33.62695	33.22727	36.04683	40.79615	47.38285	56.72114	70.80082	94.32167	141.41477	282.76250

$\Delta_4(\xi) = 0$

n \ k	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	5.31756	5.31756	5.31735	5.31298	5.28210	5.17523	4.96965	4.71085	4.45074	4.21234
1	9.28240	9.28235	9.27339	9.15259	8.85259	8.83644	9.58236	11.59433	16.36888	31.71153
2	12.68191	12.68141	12.61018	12.24157	12.50097	13.89228	16.58998	21.51241	31.74843	62.97991
3	15.96411	15.96134	15.71267	15.64653	17.04913	19.73536	24.15224	31.79591	47.34583	94.34651
4	19.19603	19.18551	18.75480	19.55120	21.94613	25.79784	31.85931	42.17317	62.99841	125.73774
5	22.40103	22.37019	21.99309	23.70425	26.98064	31.94835	39.62487	52.58819	78.67308	157.13886
6	25.58976	25.51569	25.46419	27.97537	32.08282	38.14297	47.41981	63.02216	94.35884	188.54491
7	28.76784	28.61835	29.08547	32.31086	37.22339	44.36284	55.23158	73.46701	110.05094	219.95380
8	31.93854	31.68904	32.79478	36.68550	42.38788	50.59822	63.05392	83.91857	125.74700	251.36442
9	35.10392	34.74016	36.55836	41.08580	47.56828	56.84473	70.88322	94.37469	141.44571	282.77642

$\Delta_5(\xi) = 0$

n \ k	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	6.41565	6.41565	6.41563	6.41471	6.40311	6.24500	6.12301	5.86629	5.56194	5.26835
1	10.51932	10.51932	10.51796	10.41443	10.21837	9.98535	10.41790	12.22983	16.71450	31.86930
2	13.98706	13.98703	13.97125	13.69856	13.54274	14.57346	17.04980	21.81256	31.92576	63.05929
3	17.31279	17.31259	17.19262	16.82997	17.75361	20.20104	24.46464	31.99842	47.46467	94.39941
4	20.57549	20.57449	20.31383	20.41984	22.50000	26.15030	32.09544	42.32573	63.08768	125.77756
5	23.80357	23.79987	23.36430	24.38383	27.40612	32.23177	39.81450	52.71052	78.74460	157.17062
6	27.01030	26.99915	26.56888	28.57021	32.43810	38.37988	47.57817	63.12420	94.41846	188.57175
7	30.20284	30.11275	29.98912	32.79057	37.52843	44.56630	55.36749	73.55450	110.10204	219.97647
8	33.38544	33.28081	33.56011	37.10421	42.65510	50.77677	63.17293	83.99520	125.79172	251.38430
9	36.56077	36.41342	37.22527	41.45753	47.80603	57.00333	70.98907	94.44283	141.48540	282.79422

$\Delta_6(\xi) = 0$

n \ k	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	7.50132	7.50134	7.50134	7.50115	7.49711	7.46206	7.26543	7.01548	6.66948	6.32054
1	11.73427	11.73427	11.73408	11.72070	11.54387	11.22684	11.37049	12.82714	17.12775	32.06113
2	15.26802	15.26802	15.26506	15.10890	14.75832	15.38161	17.59811	22.17418	32.14119	63.15620
3	18.63738	18.63737	18.61588	18.21929	18.60415	20.75996	24.84173	32.24427	47.60953	94.46429
4	21.93169	21.93161	21.80672	21.49687	23.12335	26.57612	32.38191	42.51147	63.19666	125.82619
5	25.18391	25.18355	24.92255	25.21973	27.92119	32.57546	40.04514	52.85966	78.83188	157.20956
6	28.40977	28.40847	27.97495	29.22246	32.86909	38.66782	47.77107	63.24871	94.49127	188.60390
7	31.61787	31.61389	31.15453	33.37547	37.89904	44.81392	55.53320	73.66134	110.16445	220.00455
8	34.81339	34.80280	34.53263	37.61442	42.98020	50.99393	63.31811	84.08875	125.84639	251.40809
9	37.99964	37.97469	38.06053	41.91051	48.09556	57.19666	71.11828	94.52604	141.53410	282.81542

$\Delta_7(\xi) = 0$

n \ k	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	8.52818	8.52818	8.52818	8.52815	8.53035	8.51207	8.34729	8.13979	7.77324	7.37455
1	12.93232	12.93232	12.93230	12.92963	12.80465	12.50810	12.41477	13.55920	17.60336	32.28621
2	16.52937	16.52937	16.53116	16.47634	16.08633	16.32040	18.32885	22.59421	32.39393	63.27052
3	19.94186	19.94186	19.93791	19.67414	19.60118	21.43978	25.28065	32.53247	47.78019	94.54062
4	23.26806	23.26805	23.24193	22.77787	23.85654	27.07221	32.71747	42.72997	63.32521	125,88342
5	26.54504	26.54501	26.41712	26.22487	28.52229	32.97775	40.31614	53.03530	78.93494	157.25541
6	29.79075	29.79063	29.53213	30.02628	33.37369	39.00576	47.99813	63.39553	94.57728	188.64193
7	33.01518	33.01685	32.58714	34.06408	38.33391	45.10509	55.72844	73.78741	110.23828	220.03730
8	36.22438	36.22446	35.74652	38.21470	43.36222	51.24957	63.48930	84.19917	125.91093	251.43713
9	39.42228	39.41918	39.09105	42.44366	48.43610	57.42443	71.27068	94.62426	141.59147	282.84119

$\Delta_8(\xi) = 0$

n \ k	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	9.59581	9.59581	9.59581	9.59581	9.59543	9.58985	9.52458	9.26081	8.87557	8.42639
1	14.11541	14.11451	14.11541	14.11729	14,07608	13.78943	13.52760	14.37244	18.13837	32.54412
2	17.77401	17.77401	17.77393	17.75576	17.44960	17.34098	18.93491	23.07056	32,68324	63.40220
3	21.22907	21.22907	21.23067	21.07632	20.73118	22.13215	25.77957	32.86214	47.97633	94.62877
4	24.58720	24.58720	24.58173	24.19986	24.78305	27.63720	33.10106	42.98080	63.47324	125.94973
5	27.88928	27.88927	27.85966	27.40058	29.22874	33.43755	40.62688	53.23736	79.05370	157.30860
6	31.15533	31.15532	31.02580	30.99977	33.95150	39.39298	48.25889	63.56458	94.67636	188.68666
7	34.39663	34.39659	34.14235	34.85442	38.83241	45.43924	55.95296	73.93263	110.32328	220.07504
8	37.62008	37.61992	37.20018	38.92314	43.80062	51.54330	63.68634	84.32645	125.98544	251.47070
9	40.83018	40.83235	40.34370	43.05818	48.82728	57.68636	71.44615	94.73748	141.65768	282.87043

$\Delta_9(\xi) = 0$

n \ k	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	10.65808	10.65808	10.65808	10.65808	10.65796	10.65848	10.61280	10.36868	9.97542	9.47824
1	15.28660	15.28659	15.28659	15.26834	15.27005	15.04807	14.68891	15.34073	18.72746	32.83402
2	19.00458	19.00458	19.00457	19.00004	18.78937	18.46671	19.70005	23.59981	33.00815	63.55112
3	22.50140	22.50140	22.50124	22.44752	21.98444	22.93872	26.33540	33.23205	48.19770	94.72863
4	25.89128	25.89128	25.89263	25.64732	25.69500	28.26800	33.53115	43.26341	63.64061	126.02472
5	29.21857	29.21857	29.21174	28.74320	29.97701	33.95296	40.97652	53.46548	79.18808	157.36862
6	32.50525	32.50525	32.47302	32.08337	34.60067	39.82825	48.55295	63.75560	94.78857	188.73626
7	35.76379	35.76379	35.63368	35.81152	39.39309	45.81559	56.20644	74.09686	110.41956	220.11816
8	39.00190	39.00189	38.75294	39.68879	44.29431	51.87455	63.90897	84.47046	126.06977	251.50790
9	42.22464	42.22458	41.81395	43.75401	49.26827	57.98204	79.76114	94.86566	141.73268	282.90414

$\Delta_{10}(\xi) = 0$

n \ k	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	11.71594	11.71595	11.71595	11.71594	11.71591	11.71423	11.68836	11.46351	11.07246	10.53004
1	16.44767	16.44767	16.44767	16.44761	16.44246	16.27354	15.88091	16.23428	19.36610	33.15495
2	20.22302	20.22301	20.22301	20.22451	20.07983	19.66846	20.53696	24.17892	33.36766	63.71714
3	23.76072	23.76072	23.76069	23.74075	23.31729	23.90103	26.94536	33.64098	48.44391	94.84003
4	27.18202	27.18202	27.18175	27.03833	26.75734	28.96216	34.00621	43.57721	63.82712	126.10838
5	30.53451	30.53451	30.53559	30.18108	30.82875	34.52225	41.36418	53.71935	79.33800	157.43561
6	33.84197	33.84197	33.83400	33.33481	35.32011	40.31042	48.87972	63.96850	94.91384	188.79240
7	37.11800	37.11800	37.08387	36.81437	40.01486	46.23334	56.48853	74.28004	110.52711	220.16617
8	40.37107	40.37107	40.24117	40.56732	44.84234	52.24278	64.15696	84.63109	126.16393	251.55002
9	43.60677	43.60676	43.36375	44.52941	49.75831	58.31107	71.86576	95.00873	141.81650	282.94103

$\Delta_{11}(\xi) = 0$

n \ k	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	12.77009	12.77010	12.77010	12.77010	12.77009	12.76932	12.75478	12.60944	12.16639	11.58176
1	17.60005	17.60006	17.60006	17.60004	17.60100	17.50975	17.08930	17.20210	20.05001	33.50622
2	21.43083	21.43083	21.43083	21.43031	21.37283	20.92092	21.51504	24.80488	33.76070	63.90007
3	25.00852	25.00852	25.00851	25.00315	24.67699	24.82253	27.60674	34.08765	48.71470	94.96297
4	28.46086	28.46086	28.46081	28.40938	27.93805	29.73125	34.52471	43.92159	64.03262	126.20085
5	31.83843	31.83843	31.83802	31.61789	31.81489	35.14373	41.78896	53.99861	79.50338	157.50974
6	35.16672	35.16672	35.16754	34.71775	36.10525	40.83828	49.23863	64.20297	95.05207	188.85406
7	38.46039	38.46039	38.45148	37.98094	40.69675	46.69165	56.79889	74.48199	110.64585	220.21893
8	41.72863	41.72863	41.69318	41.58437	45.44381	52.64738	64.43008	84.80834	126.26793	251.59612
9	44.97753	44.97753	44.87616	45.38624	50.29662	58.67300	72.10949	95.16663	141.90900	282.98275