

An Application of Cyclic Reduction to Ritz Type Difference Equations

By A. K. Rigler

1. Introduction. Iterative methods are often preferred over direct methods for solving the large systems of linear algebraic equations which arise in the finite difference approximation of boundary value problems for elliptic partial differential equations. An important reason for this preference is that the nonzero elements of the coefficient matrix are quite sparse, occurring in narrow bands along and parallel to the main diagonal. Since an iterative method such as successive over-relaxation leaves the coefficient matrix unchanged, it imposes a comparatively modest requirement on computer storage capacity.

A procedure introduced by Schröder [4] reduces the number of unknowns and improves the convergence rate of relaxation methods applied to the reduced problem. Called "cyclic reduction" by Varga [5] and "decomposition" by Collatz [1], its essential feature is the transformation of the coefficient matrix to a block triangular form. Hageman [3] proves that a block Gauss-Seidel solution of the reduced system must converge in fewer steps than a block Gauss-Seidel solution of the original. The technique is quite fruitful in improving the efficiency of finite difference solutions of potential and diffusion type problems. However, in solving some problems, for example equations with mixed derivatives, the reduction may be impractical for the following reason. The sparseness of the original coefficient matrix does not necessarily imply that the reduced coefficient matrix will be sparse; possibly the reduction would increase both the number of coefficients to be stored and the number of arithmetic operations required to complete the solution.

It is the purpose of this paper to show that equations derived for self-adjoint second order elliptic systems by using the Ritz method as described by Friedrichs [2] can be reduced without these adverse side effects.

2. Reduction of the Coefficient Matrix. Let the original set of linear algebraic equations be partitioned in the form

$$(1) \quad \begin{bmatrix} D_1 & -B \\ -B^t & D_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}.$$

Equation (1) is multiplied by the matrix

$$\begin{bmatrix} I_1 & 0 \\ B^t D_1^{-1} & I_2 \end{bmatrix}$$

to give

$$\begin{bmatrix} D_1 & -B \\ 0 & D_2 - B^t D_1^{-1} B \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 + B^t D_1^{-1} g_1 \end{bmatrix}.$$

The reduced problem is

$$(2) \quad (D_2 - B^t D_1^{-1} B) u_2 = g_2 + B^t D_1^{-1} g_1.$$

It can be shown [3] that the coefficient matrix of (2) is symmetric and positive definite whenever the coefficient matrix of (1) is symmetric and positive definite.

The usefulness of (2) in practical computation depends upon the nature of D_1^{-1} . The matrices D_1 , D_2 , and B will consist of a single band each of nonzero entries. In Hageman's examples it is pointed out that $B^t B$ is also a sparse matrix. However, D_1^{-1} is not necessarily sparse so that $B^t D_1^{-1} B$ may have no zero entries at all. Of course, if D_1 is diagonal, the reduction is profitable since the reduced coefficient matrix will certainly be sparse. In this case, the reduction is equivalent to the use of a more accurate difference formula applied to a coarser mesh.

3. Derivation of the Difference Equations. Self-adjoint elliptic equations are often solved by a direct attack on an associated variational problem. For second order equations, one can apply the Ritz method, restricting the trial function to be continuous and linear in triangles. Friedrichs [2] makes use of this method and imposes the additional requirement that the triangles be oriented in a special way.

A rectangular mesh, not necessarily uniform, is placed over the region. The mesh points are separated into two classes in the manner of a checkerboard. Each mesh rectangle is divided into two triangles by the diagonal connecting the "black" corners. Thus each triangle has its 90° vertex at a "red" point while the acute angle vertices are at black points. This is illustrated in Figure 1.

With the triangle oriented as in Figure 2, the trial function within that triangle is

$$(3) \quad u = u_{i,j} + \left(\frac{u_{i+1,j} - u_{i,j}}{h} \right) (x - x_{i,j}) + \left(\frac{u_{i,j+1} - u_{i,j}}{k} \right) (y - y_{i,j}),$$

where i, j, h, k are identified in the figure.

The integrand of the integral to be minimized is a quadratic form in u , u_x , and u_y so that over each triangle the integral has no more than three parameters to adjust; in Figure 2 they are $u_{i,j}$, $u_{i+1,j}$, and $u_{i,j+1}$.

It is evident from Figure 1, that the unknowns at red points are coupled only to unknowns at black points while unknowns at black points are coupled to both red and black unknowns. By identifying in (1) the partitions 1 and 2 with red and black points respectively, one can be assured that the reduction can profitably be applied. D_1 is a strictly diagonal matrix when only one equation is involved and when derived from a system of m equations D_1 is the direct sum of $m \times m$ blocks. Hence D_1^{-1} has the same simple form as D_1 .

4. An Example. To illustrate the method of reduction in a situation where more

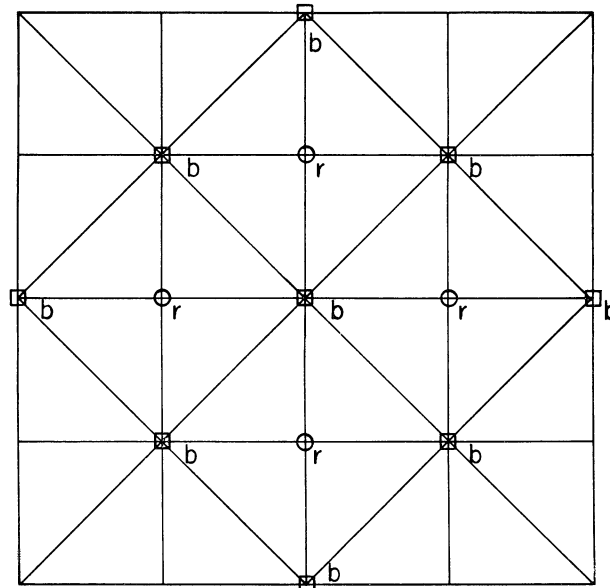


FIG. 1

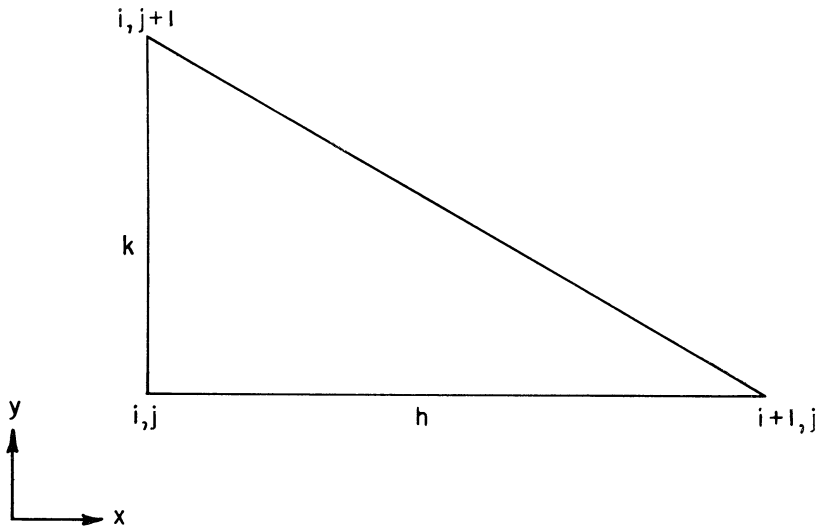


FIG. 2

conventional finite difference equations cannot be reduced, a differential operator which includes a mixed derivative was chosen. Let

$$L(u) = u_{xx} + 2bu_{xy} + u_{yy}$$

be the differential operator, where b is constant and $b^2 < 1$. On a uniform mesh,

Taylor's series might be used to generate the difference stencil

$-\frac{b}{2}$	1	$\frac{b}{2}$	u
1	-4	1	
$\frac{b}{2}$	1	$-\frac{b}{2}$	

as an approximation to $h^2L(u)$ to be applied at each mesh point.

The quadratic portion of the variational integral corresponding to $L(u)$ is

$$J(u) = \frac{1}{2} \iint (u_x^2 + 2bu_xu_y + u_y^2) dx dy.$$

Application to $J(u)$ of the Ritz method as described in §3 produces two distinct Ritz type difference equations. The stencils are

(4)

	1		u
1	-4	1	
	1		

to be applied at red points and

(5)

$-b$	1	b	u
1	-4	1	
b	1	$-b$	

to be applied at black points.

The reduced equations (3) are exactly those produced by applying the stencil

(6)

		$\frac{1}{4}$			u
	$\frac{1}{2} - b$		$\frac{1}{2} + b$		
$\frac{1}{4}$		-3		$\frac{1}{4}$	
	$\frac{1}{2} + b$		$\frac{1}{2} - b$		
		$\frac{1}{4}$			

at black points only.

It is interesting to note that neither (4) nor (5) are adequate finite difference representations of $h^2L(u)$ but the result of reduction (6) is a satisfactory representation of $2h^2L(u)$.

In conclusion, it is emphasized that the reduction procedure is always successful when applied to Ritz type difference equations; no claims are made about the relative merits of Ritz type difference equations compared to conventional equations in their accuracy in approximating the true solution.

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A Two Parameter Test Matrix

By P. J. Eberlein

1. Introduction. The $(N + 1) \times (N + 1)$ matrix A given by (1) below arose from a problem in the chemical theory of gases [1], the physically significant cases occurring when the parameter $s = 0, 1, 2, \dots$. Since the eigenvalues* and eigenvectors of A are found explicitly, the matrix is of interest in itself as a test matrix for eigenvalue programs, especially for negative real s : when $s = -2, -3, \dots -2N$, the matrix is defective with two or more pairs of eigenvectors coalescing; elsewhere in the range $-2N < s < -2$ at least one pair of eigenvectors is nearly parallel. In this range, the positive roots of the characteristic polynomial are ill-conditioned, especially for $s < -(N + 1)$.

The matrix, its eigenvalues, and its right and left eigenvectors are given in section 2; a few numerical experiments are described in section 3.

2. The Matrix. Let

$$(1) \quad A = (a_{ij}) = \begin{pmatrix} -N & N + s & 0 & 0 & \vdots \\ N & -(3N + s - 2) & 2(N + s - 1) & 0 & \\ 0 & 2(N - 1) & -(5N + 2s - 8) & 3(N + s - 2) & 0 \\ 0 & 0 & 3(N - 2) & \cdot & \cdot \\ \dots & & & & \cdot \end{pmatrix}$$

Received February 1, 1963, revised November 10, 1963.

* Conjectured by Brauner and Wilson.