A Note on Some Quadrature Formulas for the Interval $(-\infty, \infty)$

By Seymour Haber

In a paper in this journal [1], W. M. Harper proposed a family of "Gaussian" quadrature formulas for $\int_{-\infty}^{\infty} (1 + x^2)^{-k-1} f(x)\, dx$. It is the purpose of this note to re-derive some of his formulas from a different point of view, which suggests a different manner of using them and leads to a convergence theorem

For the approximate evaluation of the integral over $(-\infty, \infty)$ of a rational or algebraic integrand—or any integrand which goes to zero as a negative power of $|x|$ as x goes to infinity—it seems reasonable to want a formula based on a weight function with similar behavior, rather than on e^{-x^2} as in the Hermite-Gauss quadrature. If the integrand f goes to zero as $|x|^{-p}$, a natural choice of weight function is $w_\alpha(x) = (1 + x^2)^{-\alpha}$, $\alpha = p/2$. Setting $f(x) = w_\alpha(x) g(x)$, we are led to consider formulas of the form:

$$\int_{-\infty}^{\infty} w_\alpha(x) g(x)\, dx \sim \sum_{i=1}^{n} A_i^{(\alpha)} g(x_i^{(\alpha)}); \qquad \alpha > \frac{1}{2} \tag{1}$$

where g is a bounded function.

Since g is bounded on the whole line, we cannot base the formulas on consideration of polynomial approximation to g; one choice that suggests itself is to consider approximation to g by functions of the form

$$a_0 + \frac{a_1 + b_1 x}{1 + x^2} + \frac{a_2 + b_2 x}{(1 + x^2)^2} + \cdots + \frac{a_m + b_m x}{(1 + x^2)^m}.$$

We thus look to determine the abscissas and coefficients of the quadrature formula so as to make it exact for all functions of this form for m as high as possible.

Since $w_\alpha(x)$ is even, requiring that the formula be symmetric about zero, i.e. of the form

$$2B_0^{(\alpha)} g(0) + \sum_{i=1}^{N} B_i^{(\alpha)} [g(x_i^{(\alpha)}) + g(-x_i^{(\alpha)})]$$

insures its exactness for all term $b_r x(1 + x^2)^{-r}$, and only the terms a_0, $a_1(1 + x^2)^{-1}$, $a_2(1 + x^2)^{-2}$, $\cdots$ need further consideration. These are all even, and so it amounts to the same thing to consider the quadrature formula

$$\int_0^{\infty} w_\alpha(x) g(x)\, dx \sim B_0^{(\alpha)} g(0) + \sum_{i=1}^{N} B_i^{(\alpha)} g(x_i^{(\alpha)}). \tag{2}$$

The $B_i^{(\alpha)}$ and $x_i^{(\alpha)}$ are to be determined so as to maximize the highest integer M such that (2) is exact whenever $g = P((1 + x^2)^{-1})$ with P a polynomial of degree M or lower.

For such g, setting $y = (1 + x^2)^{-1}$ transforms the integral in (2) into $\frac{1}{2} \int_0^1 y^{\alpha - 3/2} (1 - y)^{-1/2} P(y)\, dy$; and the Jacobi-Gauss quadrature formula (see [2]

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[3]) for the exponents $\alpha - \frac{3}{2}$ and $-\frac{1}{2}$ and for N abscissas evaluates this last integral exactly whenever the degree of P is $\leqq 2N - 1$, and that is the best that can be done. Thus our abscissas and coefficients are given by (since all the $y_i^{(\alpha)}$ are less than 1):

$$(3) \qquad B_0^{(\alpha)} = 0; \qquad B_i^{(\alpha)} = \tfrac{1}{2}C_i^{(\alpha)}, \qquad x_i^{(\alpha)} = (1 - y_i^{(\alpha)})^{1/2}(y_i^{(\alpha)})^{-1/2}, \quad i \geqq 1$$

where the $C_i^{(\alpha)}$ and $y_i^{(\alpha)}$ are the coefficients and abscissas of the Jacobi-Gauss formula.

Since the set of all functions of the form

$$(1 + x^2)^{-\alpha}\left[a_0 + \frac{a_1 + b_1 x}{(1 + x^2)} + \cdots + \frac{a_{2N-1} + b_{2N-1}x}{(1 + x^2)^{2N-1}}\right]$$

is also that of all functions of the form $(1 + x^2)^{-2N-\alpha+1}Q(x)$ where Q is a polynomial of degree $4N - 2$ or lower, the conditions determining the above formula for any α and N are the same as those determining Harper's formula for (using "k" and "n" in the meaning given them in [1]) $k = \alpha + 2N - 2$, $n = 2N$. Thus we have just re-derived Harper's formulas for even n.

It follows from known properties of Jacobi-Gauss quadrature that the coefficients are non-negative; and if f is continuous and α is chosen large enough to make g bounded, it follows that the approximation obtained converges to the integral as N increases.

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1. W. M. Harper, "Quadrature formulas for infinite integrals," *Math. Comp.*, v. 16, 1962, p. 170-175.
2. V. I. Krylov, *Approximate Calculation of Integrals*, Macmillan, New York, 1962, Chapter 7.
3. F. B. Hildebrand, *Introduction to Numerical Analysis*, McGraw-Hill, New York, 1956, p. 331–334.

Generalized Trigonometric Functions

By F. D. Burgoyne

In an investigation into geometrical properties of the curves $x^n/a^n + y^n/b^n = 1$, use was made of the functions $s_n(u)$ where

$$u = \int_0^{s_n(u)} (1 - t^n)^{1/n-1}\, dt \qquad (0 \leqq u \leqq P_n)$$

and

$$P_n = \int_0^1 (1 - t^n)^{1/n-1}\, dt = 2\left\{\left(\frac{1}{n}\right)!\right\}^2 \Big/ \left(\frac{2}{n}\right)!\,.$$

These functions may be called generalized trigonometric functions in view of the fact that $s_2(u) = \sin u$. Further, $s_3(u)$ is the Dixon function smu, considered by Dixon [1], Adams [2], and Laurent [3]. For $n = 4$ and 6 the functions are related to the Jacobian elliptic functions $sn(u)$ with moduli $2^{1/2}/2$, $(2 - 3^{1/2})^{1/2}/2$

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