[3]) for the exponents $\alpha - \frac{3}{2}$ and $-\frac{1}{2}$ and for N abscissas evaluates this last integral exactly whenever the degree of P is $\leqq 2N - 1$, and that is the best that can be done. Thus our abscissas and coefficients are given by (since all the $y_i^{(\alpha)}$ are less than 1):

$$(3) \quad B_0^{(\alpha)} = 0; \qquad B_i^{(\alpha)} = \tfrac{1}{2}C_i^{(\alpha)}, \qquad x_i^{(\alpha)} = (1 - y_i^{(\alpha)})^{1/2}(y_i^{(\alpha)})^{-1/2}, \quad i \geqq 1$$

where the $C_i^{(\alpha)}$ and $y_i^{(\alpha)}$ are the coefficients and abscissas of the Jacobi-Gauss formula.

Since the set of all functions of the form

$$(1 + x^2)^{-\alpha}\left[a_0 + \frac{a_1 + b_1 x}{(1 + x^2)} + \cdots + \frac{a_{2N-1} + b_{2N-1}x}{(1 + x^2)^{2N-1}}\right]$$

is also that of all functions of the form $(1 + x^2)^{-2N-\alpha+1}Q(x)$ where Q is a polynomial of degree $4N - 2$ or lower, the conditions determining the above formula for any α and N are the same as those determining Harper's formula for (using "k" and "n" in the meaning given them in [1]) $k = \alpha + 2N - 2$, $n = 2N$. Thus we have just re-derived Harper's formulas for even n.

It follows from known properties of Jacobi-Gauss quadrature that the coefficients are non-negative; and if f is continuous and α is chosen large enough to make g bounded, it follows that the approximation obtained converges to the integral as N increases.

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Generalized Trigonometric Functions

By F. D. Burgoyne

In an investigation into geometrical properties of the curves $x^n/a^n + y^n/b^n = 1$, use was made of the functions $s_n(u)$ where

$$u = \int_0^{s_n(u)} (1 - t^n)^{1/n-1}\, dt \qquad (0 \leqq u \leqq P_n)$$

and

$$P_n = \int_0^1 (1 - t^n)^{1/n-1}\, dt = 2\left\{\left(\frac{1}{n}\right)!\right\}^2 \Big/ \left(\frac{2}{n}\right)!\,.$$

These functions may be called generalized trigonometric functions in view of the fact that $s_2(u) = \sin u$. Further, $s_3(u)$ is the Dixon function smu, considered by Dixon [1], Adams [2], and Laurent [3]. For $n = 4$ and 6 the functions are related to the Jacobian elliptic functions $sn(u)$ with moduli $2^{1/2}/2$, $(2 - 3^{1/2})^{1/2}/2$

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TABLE 1

n	P_n
3	1.76664
4	1.85407
5	1.90030
6	1.92762

TABLE 2

Values of generalized trigonometric functions

u	$s_3(u)$	δ^2	$s_4(u)$	δ^2	$s_5(u)$	δ^2	$s_6(u)$	δ^2
0.00	0.00000	0	0.00000	0	0.00000	0	0.00000	0
0.05	0.05000	−2	0.05000	0	0.05000	0	0.05000	0
0.10	0.09998	4	0.10000	−1	0.10000	0	0.10000	0
0.15	0.14992	13	0.14999	3	0.15000	−1	0.15000	0
0.20	0.19973	19	0.19995	6	0.19999	1	0.20000	−1
0.25	0.24935	31	0.24985	11	0.24997	5	0.24999	1
0.30	0.29866	45	0.29964	21	0.29990	7	0.29997	3
0.35	0.34752	60	0.34922	33	0.34976	16	0.34992	6
0.40	0.39578	76	0.39847	46	0.39946	26	0.39981	14
0.45	0.44328	95	0.44726	66	0.44890	40	0.44956	24
0.50	0.48983	116	0.49539	89	0.49794	61	0.49907	38
0.55	0.53522	135	0.54263	115	0.54637	88	0.54820	61
0.60	0.57926	157	0.58872	143	0.59392	118	0.59672	93
0.65	0.62173	175	0.63338	176	0.64029	158	0.64431	133
0.70	0.66245	197	0.67628	205	0.68508	199	0.69057	182
0.75	0.70120	212	0.71713	237	0.72788	246	0.73501	240
0.80	0.73783	230	0.75561	266	0.76822	288	0.77705	300
0.85	0.77216	242	0.79143	289	0.80568	327	0.81609	357
0.90	0.80407	253	0.82436	307	0.83987	360	0.85156	405
0.95	0.83345	260	0.85422	320	0.87046	377	0.88298	435
1.00	0.86023	264	0.88088	322	0.89728	383	0.91005	445
1.05	0.88437	264	0.90432	319	0.92027	374	0.93267	429
1.10	0.90587	260	0.92457	307	0.93952	353	0.95100	395
1.15	0.92477	254	0.94175	289	0.95524	321	0.96538	346
1.20	0.94113	243	0.95604	266	0.96775	283	0.97630	291
1.25	0.95506	230	0.96767	238	0.97743	239	0.98431	233
1.30	0.96669	214	0.97692	208	0.98472	197	0.98999	179
1.35	0.97618	196	0.98409	179	0.99004	158	0.99388	134
1.40	0.98371	177	0.98947	148	0.99378	120	0.99643	95
1.45	0.98947	153	0.99337	119	0.99632	91	0.99803	65
1.50	0.99370	131	0.99608	92	0.99795	63	0.99898	41
1.55	0.99662	108	0.99787	70	0.99895	44	0.99952	27
1.60	0.99846	83	0.99896	48	0.99951	27	0.99979	14
1.65	0.99947	58	0.99957	32	0.99980	15	0.99992	7
1.70	0.99990	33	0.99986	18	0.99994	10	0.99998	−5
1.75	1.00000	−9	0.99997	8	0.99998	2	0.99999	0
1.80			1.00000	−3	1.00000	−2	1.00000	−1
1.85			1.00000	0	1.00000	0	1.00000	0
1.90					1.00000	0	1.00000	0

respectively. (See Byrd and Friedman [4] p. 158.) General properties of these functions are discussed in some detail by Shelupsky [5].

Tabulations of $s_3(u)$ are given in [2] to four decimal places for $u = 0(P_3/120)P_3$ and in [3] to ten decimal places for $u = 0(0.001)0.103$, but no direct tabulation of $s_n(u)$ for $n \geqq 4$ is known to the author. For this reason it was decided to tabulate $s_n(u)$ for $n = 4, 5, 6$, and it was considered convenient to tabulate $s_3(u)$ also.

In Table 2 $s_n(u)$ is given to five decimal places for $n = 3, 4, 5, 6$ and $u = 0(0.05)P_n^*$, where $P_n^* \leqq P_n < P_n^* + 0.05$. Second differences are given alongside the tabular values, thus permitting interpolation at non-tabular points by means of Everett's interpolation formula

$$f_p = (1 - p)f_0 + pf_1 + E_2\delta_0^2 + F_2\delta_1^2$$

where

$$E_2 = -p(p - 1)(p - 2)/6$$

and

$$F_2 = (p + 1)p(p - 1)/6.$$

Fourth differences are everywhere sufficiently small to ensure that the error due to to interpolation will be less than 0.54 units in the fifth decimal place. The tabulation was performed on a Mercury computer, a fourth-order Runge-Kutta process being applied to the differential equation

$$s_n'(u) = \{1 - s_n^n(u)\}^{1-1/n}$$

starting from $s_n(0) = 0$. In Table 1 we tabulate P_n for $n = 3, 4, 5, 6$. The functions

$$c_n(u) = \{1 - s_n^n(u)\}^{1/n}$$

may be evaluated from these tables by means of the relation

$$c_n(u) = s_n(P_n - u).$$

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