this to

$$
\begin{aligned}
a_n(k) &= (-1)^n 2 \int_0^\infty e^{-t^2} J_{2n+1}(2kt)\,dt \\
&= (-1)^n \sqrt{\pi} e^{-k^2/2} I_{n+(1/2)}(k^2/2) \qquad (5) \\
&= \sum_{r=0}^{n} \frac{(n+r)!}{r!(n-r)!} k^{-2r-1}[(-1)^{r+n} - e^{-k^2}], \qquad n = 0, 1, 2 \cdots.
\end{aligned}
$$

This expression may easily be seen to be consistent with (4).

Acknowledgments. This work was supported by the Office of Naval Research.

Department of Physics,
University College London,
Gower Street,
London, W. C. 1.

1. D. Harris III, "On the line-absorption coefficient due to Doppler effect and damping," *Astrophys. J.*, v. 108, 1948, p. 112–115.
2. D. G. Hummer, "Noncoherent scattering. I. The redistribution functions with Doppler broadening," *Monthly Notices Roy. Astronom. Soc.*, v. 125, 1963, p. 21–37.
3. W. L. Miller & A. R. Gordon, "Numerical evaluation of infinite series and integrals which arise in certain problems of linear heat flow, electrochemical diffusion, etc.," *J. Chem. Phys.*, v. 35, 1935, p. 2785–2884.
4. J. B. Rosser, *Theory and Application of* $\int_0^\infty e^{-x^2}\,dx$ *and* $\int_0^z e^{-p^2y^2}\,dy \int_0^y e^{-x^2}\,dx$, Mapleton House, Brooklyn, N. Y., 1948.
5. B. Lohmander & S. Rittsten, "Tables of the function $y = e^{-x^2} \int_0^x e^{t^2}\,dt$," *Kungl. Fysiogr. Sällsk. i Lund Förh.*, v. 28, 1958, p. 45–52.
6. H. M. Terrill & L. Sweeny, "An extension of Dawson's table of the integral of e^{x^2}," *J. Franklin Inst.*, v. 237, 1944, p. 495–497; "Table of the integral of e^{x^2}," *ibid.*, v. 238, 1944, p. 220–222.
7. National Physical Laboratory, *Modern Computing Methods*, 2nd edition, H. M. Stationery Office, London, 1961.
8. C. W. Clenshaw, "A note on the summation of Chebyshev series," *MTAC*, v. 9, 1955, p. 118–120; see also [7], Chapter 8.

First One Hundred Zeros of $J_0(x)$ Accurate to 19 Significant Figures

By Henry Gerber

1. Introduction. Some physical investigations require a knowledge of accurate values of the zeros of the Bessel function $J_0(x)$. The most extensive values previously published are those of the British Association for the Advancement of Science [1], which consist of 10 decimal places. More accurate values have now been computed, and are presented in Table 1. The minimum accuracy of the tabulated zeros is 19 significant figures.

2. Method of Computation. Two methods were used to compute the roots. The first twelve roots were computed by the method of "false position." The values of

Received August 12, 1963.

TABLE 1

The first one hundred roots of $J_0(x) = 0$

(n = Number of Zero)

n	x_n	n	x_n
1	2.40482 55576 95772 769	51	159.43661 11642 63146 32
2	5.52007 81102 86310 649	52	162.57818 86689 46677 52
3	8.65372 79129 11012 216	53	165.71976 67479 55020 87
4	11.79153 44390 14281 615	54	168.86134 53692 35825 69
5	14.93091 77084 87785 948	55	172.00292 45030 78200 21
6	18.07106 39679 10922 545	56	175.14450 41219 02743 06
7	21.21163 66298 79258 960	57	178.28608 42000 73770 68
8	24.35247 15307 49302 736	58	181.42766 47137 31050 79
9	27.49347 91320 40254 79	59	184.56924 56406 38718 14
10	30.63460 64684 31975 12	60	187.71082 69600 49359 78
11	33.77582 02135 73568 69	61	190.85240 86525 81522 32
12	36.91709 83536 64043 98	62	193.99399 07001 09119 79
13	40.05842 57646 28239 29	63	197.13557 30856 61414 74
14	43.19979 17131 76730 36	64	200.27715 57933 32411 78
15	46.34118 83716 61814 02	65	203.41873 88081 98646 17
16	49.48260 98973 97817 17	66	206.56032 21162 44473 65
17	52.62405 18411 14996 03	67	209.70190 57042 94075 20
18	55.76551 07550 19979 31	68	212.84348 95599 49482 75
19	58.90698 39260 80942 13	69	215.98507 36715 34013 16
20	62.04846 91902 27169 88	70	219.12665 80280 40567 46
21	65.18996 48002 06860 44	71	222.26824 26190 84314 34
22	68.33146 93298 56798 27	72	225.40982 74348 59329 90
23	71.47298 16035 93732 82	73	228.55141 24660 98813 30
24	74.61450 06437 01837 88	74	231.69299 77040 38538 78
25	77.75602 56303 88055 04	75	234.83458 31403 83241 02
26	80.89755 58711 37627 86	76	237.97616 87672 75662 86
27	84.03909 07769 38190 16	77	241.11775 45772 68022 51
28	87.18062 98436 41153 65	78	244.25934 05632 95682 56
29	90.32217 26372 10480 06	79	247.40092 67186 52824 85
30	93.46371 87819 44774 17	80	250.54251 30369 69955 47
31	96.60526 79509 96268 78	81	253.68409 95121 93081 00
32	99.74681 98586 80596 47	82	256.82568 61385 64413 02
33	102.88837 42541 94794 60	83	259.96727 29106 04471 57
34	106.02993 09164 51615 51	84	263.10885 98230 95470 69
35	109.17148 96498 05383 55	85	266.25044 68710 65880 12
36	112.31305 02804 94909 63	86	269.39203 40497 76067 14
37	115.45461 26536 66939 63	87	272.53362 13547 04931 45
38	118.59617 66308 72531 72	88	275.67520 87815 37453 85
39	121.73774 20879 50962 96	89	278.81679 63261 53086 58
40	124.87930 89132 32946 04	90	281.95838 39846 14919 85
41	128.02087 70060 08324 08	91	285.09997 17531 59564 54
42	131.16244 62752 13914 61	92	288.24155 96281 87696 44
43	134.30401 66383 05466 10	93	291.38314 76062 55212 24
44	137.44558 80202 84277 79	94	294.52473 56840 64951 46
45	140.58716 03528 54296 55	95	297.66632 38584 58942 52
46	143.72873 35736 89732 53	96	300.80791 21264 11134 77
47	146.87030 76257 96649 59	97	303.94950 04850 20581 11
48	150.01188 24569 54757 49	98	307.09108 89315 05039 11
49	153.15345 80192 27892 49	99	310.23267 74631 94960 95
50	156.29503 42685 33523 82	100	313.37426 60775 27844 72

$J_0(x)$ corresponding to a given trial root x were calculated by direct interpolation of the Harvard tables [2], which give $J_0(x)$ accurate to 18 decimal places. For $0 \leqq x \leqq 25$ the argument increment h is 0.001; for $25 < x \leqq 100$ the increment is 0.01. Seven terms of the Newton-Bessel central difference formula [3] were used in the interpolation. This formula requires eight tabulated values of $J_0(x_0 + mh)$, where

$$x_0 = \text{greatest tabulated argument not exceeding } x$$

$$m = \pm 1, \pm 2, \pm 3, -4.$$

This method of computation has two advantages. First, in the vicinity of a zero of $J_0(x)$ the tabulated values consist of only 14 to 16 significant figures. The double-precision method of programming the IBM 7090 computer permits calculations with 17 significant digits. Thus, the above values of $J_0(x_0 + mh)$, which serve as "constants" for the interpolation process, can be entered into the computer without error.

Secondly, the interpolation variable u is given by the relationship

$$u = (x - x_0)/h \tag{1}$$

where

$$x_0 < x < x_0 + h. \tag{2}$$

The number of significant figures in the root, x, is thus equal to the sum of the number of significant figures in u and x_0. An examination of the interpolation formula shows that fewer than two significant digits are lost because of round-off error. Consequently the variable, u, can be calculated accurate to 15 significant figures. Interpolation of the Harvard tables by means of double-precision computation thus gives the roots accurate to 18 decimal places for $x \leqq 25$, and 17 decimal places for $25 < x < 100$.

The roots of $J_0(x)$ can also be computed by the following asymptotic series given by Bickley and Miller [4]. Let

$$c_n = 1/(4n - 1)\pi \qquad n = 1, 2, 3 \cdots \tag{3}$$

The nth root $j_{0,n}$ is then given by the expression

$$\begin{aligned} j_{0,n} = \left(n - \frac{1}{4}\right)\pi + \frac{c_n}{2} - \frac{31c_n^3}{6} + \frac{3779c_n^5}{15} - \frac{62\,77237c_n^7}{210} \\ + \frac{20921\,63573c_n^9}{315} - \frac{824\,97257\,36393c_n^{11}}{3465} \\ + \frac{847\,49688\,72511\,28654c_n^{13}}{6\,75675} \cdots \end{aligned} \tag{4}$$

The first one hundred roots were computed by means of Eq. (4). For n equal to or larger than 11, roots calculated by the two methods agree to 17 decimal places. This agreement confirms the validity of Eq. (4), and confirms the accuracy of the corresponding zeros in Table 1. It is interesting to note that discrepancies in the 10th decimal place of x_n occur between the data of Table 1 and the earlier tables at $n = 4, 5, 8, 41, 45, 85, 95$, and 100. These differences, which are all less than 1.2×10^{-10}, are presumably due to errors in the previous calculations.

3. Acknowledgments. The author gratefully acknowledges the suggestions of Mr. Charles R. Newman for programming the computer.

Naval Ordnance Laboratory
White Oak, Maryland

1. British Association for the Advancement of Science, *Mathematical Tables, v. 6: Bessel Functions, Part I*, Cambridge University Press, Cambridge, 1950.
2. Harvard Computation Laboratory, *Tables of the Bessel Functions of the First Kind of Orders Zero and One*, Harvard University Press, Cambridge, Massachusetts, 1947.
3. *Ibid.*, p. xix–xxii.
4. W. G. Bickley & J. C. P. Miller, *Notes on the Evaluation of Zeros and Turning Values of Bessel Functions*, *Phil. Mag.* 36, 1945, p. 121–133.

Polylogarithms, Dirichlet Series, and Certain Constants

By Daniel Shanks

The *polylogarithms* $F_s(z)$ are defined by

$$F_s(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^s} \tag{1}$$

for $|z| < 1$ and for the real part of $s \geqq 0$, and by analytic continuation for other values of z and s. They can be regarded as functions of z, with a parameter s, given by the power series (1), or as functions of s, with a parameter z, given by the Dirichlet series (1).

Recently [1] we discussed the Dirichlet series defined by

$$L_a(s) = \sum_{k=0}^{\infty} \frac{\left(\frac{-a}{2k+1}\right)}{(2k+1)^s} \tag{2}$$

and its analytic continuation, where $\left(\frac{-a}{2k+1}\right)$ is the Jacobi symbol. It is expressible in closed form for three-quarters of all combinations of integers a and s; namely, for $s \leqq 1$ and all a, for s even and >1 if $a < 0$, and for s odd and > 1 if $a > 0$.

The remaining, non-closed form $L_a(n)$ for $a = \pm 2, \pm 3$, and ± 6, with $n \leqq 10$, were computed [1] by a device, which (in essence) is based on the fact that all of the so-called *characters* modulo 8, 12, or 24 are real. In contrast, the corresponding $L_a(n)$ for $a = \pm 5, \pm 7$, and ± 10, say, which were also desired, are not obtainable by that method, unless it is modified, since now some of the characters are complex.

We did, however, express $L_a(s)$ as a linear combination of the functions $S_s(x)$ or $C_s(x)$ for various values of x determined by the integer a [1, equations (24)–(27)]. These functions [1, equation (18)] are defined by

$$\begin{aligned} S_s(x) &= \sum_{k=0}^{\infty} \frac{\sin 2\pi(2k+1)x}{(2k+1)^s}, \\ C_s(x) &= \sum_{k=0}^{\infty} \frac{\cos 2\pi(2k+1)x}{(2k+1)^s}. \end{aligned} \tag{3}$$

Received July 23, 1963.