Chebyshev Approximations of a Function and Its Derivatives

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1. Introduction. This paper considers the problem of the uniform approximation of a function and its first r derivatives. Several theorems concerning the number and nature of the extremals of a best approximation are obtained. These results are applied to a special case of approximating a function and its first derivative, and a uniqueness theorem is obtained.

2. Statement of the Problem. Let X be a compact subset of the real line. Let $n \geqq 0$ and $r \geqq 1$ be fixed integers. The function $f(x)$, which is to be approximated, and the base functions $\phi_0(x), \phi_1(x), \cdots, \phi_n(x)$ are all assumed to have continuous rth derivatives. Let $w_0(x), w_1(x), \cdots, w_r(x)$ be positive weight functions, continuous on X.

1. *Definition.* $M[g] = \max_{k=0,1,\cdots,r} \| w_k(x) D^k g(x) \|$ where $\| \quad \|$ denotes the uniform norm on X, and g has a continuous rth derivative.

2. *Problem.* Find real scalars $a_0, a_1, \cdots, a_n$ such that $M[\sum a_i\phi_i(x) - f(x)]$ is a minimum.

The function f may be given by a table or by other means. By the statement $g \equiv 0$ we shall mean that $D^k g(x) = 0$ for $k = 0, 1, \cdots, r$ and for all $x \in X$. The functional M is a norm, so that

3. $0 \leqq M[g] < \infty$.
4. $M[g] = 0$ iff $g \equiv 0$.
5. $M[tg] = | t | M[g]$ where t is any real number.
6. $M[g + h] \leqq M[g] + M[h]$.

Points in Euclidean $n + 1$ space are represented by $\alpha = (a_0, a_1, \cdots, a_n)$, $\beta = (b_0, b_1, \cdots, b_n)$ etc., while polynomials are given by $P(x, \alpha) = \sum a_i\phi_i(x)$, etc. In addition we let $e = \inf M[P(x, \alpha) - f]$ where the infimum is taken over all α in $n + 1$ space, and let $R = \{\alpha \in E^{n+1}: M[P(x, \alpha) - f] = e\}$. The base functions $\phi_0, \phi_1, \cdots, \phi_n$ are assumed to be linearly independent in the sense that $P(x, \alpha) = 0$ only if $\alpha = 0$.

Using the fact that the norm M is a continuous linear functional on E^{n+1} it follows that the set R of best approximations is closed, bounded, convex and nonempty. These are standard results, and proofs may be found in Achieser [1] and Buck [2].

3. Example. The following example was chosen to illustrate the difference between approximating a function and its derivatives, and ordinary Chebyshev approximation. Let $w_0 \equiv 1$, $w_1 \equiv 1$, $r = 1$ and suppose that f and Df are given by Table 7. The problem is to find $a_0, a_1, \cdots, a_n$ for various n, so that

$$M[a_0 + a_1x + \cdots + a_nx^n - f]$$

is a minimum.

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7. Table.

x	-1	$-1/\sqrt{3}$	0	$1/\sqrt{3}$	1
$f(x)$ $Df(x)$	1 0	0 1	-1 0	0 1	-1 0

First consider the case $n = 4$. For notational convenience let $x_1 = -1$, $x_2 = -1/\sqrt{3}$, $x_3 = 0$, $x_4 = 1/\sqrt{3}$ and $x_5 = 1$. Since $M[f] = 1$, the set R of best approximations must be contained in the region of Euclidean 5-space whose points α satisfy the relation $M[P(x, \alpha) - f] \leqq 1$. This region is defined by the inequalities

$$(1) \qquad -1 \leqq P(x_i, \alpha) - f(x_i) \leqq 1, \qquad i = 1, 2, \cdots, 5,$$

$$(2) \qquad -1 \leqq DP(x_i, \alpha) - Df(x_i) \leqq 1, \qquad i = 1, 2, \cdots, 5.$$

Consider the following subsystem of inequalities:

$$(3) \qquad a_0 - a_1 + a_2 - a_3 + a_4 \geqq 0 \qquad \text{from} \quad (1), i = 1,$$

$$(4) \qquad -a_0 - a_1 - a_2 - a_3 - a_4 \geqq 0 \qquad \text{from} \quad (1), i = 5,$$

$$(5) \qquad a_1 - 2a_2/\sqrt{3} + a_3 - 4a_4/3\sqrt{3} \geqq 0 \qquad \text{from} \quad (2), i = 2,$$

$$(6) \qquad a_1 + 2a_2/\sqrt{3} + a_3 + 4a_4/3\sqrt{3} \geqq 0 \qquad \text{from} \quad (2), i = 4.$$

From adding (4), (5), (6) and comparing the result with (3) it follows that any solution to the system (1), (2) must satisfy

$$(7) \qquad a_0 - a_1 + a_2 - a_3 + a_4 = 0,$$

$$(8) \qquad -a_0 - a_1 - a_2 - a_3 - a_4 = 0,$$

$$(9) \qquad a_1 - 2a_2/\sqrt{3} + a_3 - 4a_4/3\sqrt{3} = 0,$$

$$(10) \qquad a_1 + 2a_2/\sqrt{3} + a_3 + 4a_4/3\sqrt{3} = 0.$$

From (7), (8) we get $a_0 + a_2 + a_4 = 0$ and $a_1 + a_3 = 0$, while from (9), (10) it follows that $a_2 + 2a_4/3 = 0$. Thus polynomials satisfying (1), (2) must be of the form

$$P(x, \alpha) = s(x - x^3) + t(-\tfrac{1}{3} - 2x^2/3 + x^4),$$

where s and t remain to be determined. Table 8 gives the results thus far obtained.

8. Table.

x	-1	$-1/\sqrt{3}$	0	$1/\sqrt{3}$	1
$P - f$ $D(P - f)$	-1 $-2s - 8t/3$	$-2s/3\sqrt{3} - 4t/9$ -1	$-t/3 + 1$ s	$2s/3\sqrt{3} - 4t/9$ -1	1 $-2s + 8t/3$

It is now evident that if $P(x, \alpha)$ is a best approximation to f, then $M[P(x, \alpha) - f] = 1$. Moreover every polynomial of the form $P(x, \alpha) = s(x - x^3) +$

$t(-\frac{1}{3} - 2x^2/3 + x^4)$ is a best approximation provided

$$-1 \leqq -2s/3\sqrt{3} - 4t/9 \leqq 1, \tag{11}$$

$$-1 \leqq -t/3 + 1 \leqq 1, \tag{12}$$

$$-1 \leqq 2s/3\sqrt{3} - 4t/9 \leqq 1, \tag{13}$$

$$-1 \leqq -2(3s + 4t)/3 \leqq 1, \tag{14}$$

$$-1 \leqq s \leqq 1, \tag{15}$$

$$-1 \leqq 2(-3s + 4t)/3 \leqq 1. \tag{16}$$

The solution to this system is a triangular region in the s, t plane defined by $t \geqq 0$, $-3s + 4t \leqq \frac{3}{2}$, and $3s + 4t \leqq \frac{3}{2}$. Thus the case $n = 4$ provides an example of a two-dimensional solution space.

To solve the case $n = 3$ it is merely necessary to note that the coefficient of x^4 must be zero. Hence all solutions are given by $P(x, \alpha) = s(x - x^3)$, $-\frac{1}{2} \leqq s \leqq \frac{1}{2}$. For the cases $n = 0, 1, 2$ it is evident that the only solution is $s = 0$, $t = 0$, so that $P(x, \alpha) \equiv 0$ is the unique best approximation.

For $n > 4$ the problem can be formulated as a linear programming problem, and solved using standard techniques. Solutions for these cases are given in 10.–12., with a summary of the results in Table 9. The values of e have been rounded to four places, while the coefficients of the polynomials are given to five decimal places. The case $n = 6$ has a solution space which is at least one dimensional. The solution to $n = 5$ is a point in the solution space to $n = 6$, so these two have been combined in 10.

9. TABLE.

$n \leqq 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n \geqq 9$
$e = 1.0$	$e = .8632$	$e = .8632$	$e = .5222$	$e = .4227$	$e = 0$

10. $n = 5$, $n = 6$. $P(x) = -.13681 - .86319x + .27362x^2 + 1.81595x^3 - .13681x^4 - 1.08957x^5$.

x	-1	$-1/\sqrt{3}$	0	$1/\sqrt{3}$	1
$P - f$	$-.8632$	$.1580$	$.8632$	$-.2796$	$.8632$
$D(P - f)$	$-.8632$	$-.8632$	$-.8632$	$-.4419$	$-.8632$

11. $n = 7$. $P(x) = -.47789 - .52211x + 3.24209x^2 + 10.49572x^3 - 6.54944x^4 - 22.54194x^5 + 3.37262x^6 + 11.67781x^7$.

x	-1	$-1/\sqrt{3}$	0	$1/\sqrt{3}$	1
$P - f$	$-.5221$	$-.5222$	$.5221$	$.5222$	$-.3031$
$D(P - f)$	$-.5221$	$-.5220$	$-.5221$	$-.5220$	$.5221$

12. $n = 8$. $P(x) = -1.42265 - .42265x + 11.38120x^2 + 8.10363x^3 - 31.29829x^4 - 16.74871x^5 + 34.14359x^6 + 8.49038x^7 - 12.80385x^8$

x	-1	$-1/\sqrt{3}$	0	$1/\sqrt{3}$	1
$P - f$	$-.4226$	$-.4226$	$.4226$	$.4226$	$.4226$
$D(P - f)$	$-.4226$	$-.4226$	$-.4226$	$-.4226$	$-.4226$

4. Characterization. Instead of considering just one error function $P - f$, as in ordinary Chebyshev approximation, we must consider $r + 1$ weighted error functions.

13. *Definition.* $L_k(x, \alpha) \equiv w_k(x)D^k[P(x, \alpha) - f(x)]$. The functions $L_k(x, \alpha)$, defined for all α in Euclidean $n + 1$ space and $k = 0, 1, \cdots r$ are called *weighted error functions.*

14. *Definition.* Suppose $M[P(x, \alpha) - f] = d$. The pair (x_0, k), where $x_0 \in X$ and $0 \leqq k \leqq r$ is an integer, is called an extremal with respect to the approximation $P(x, \alpha)$ to f if $|L_k(x_0, \alpha)| = d$.

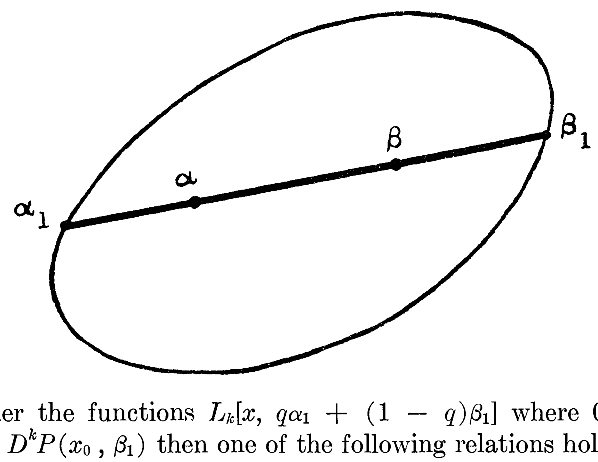
Because X is compact and the functions $L_k(x, \alpha)$ are continuous, it is evident that every approximation has one or more extremals. We shall now establish some results concerning the extremals of a best approximation.

15. *Definition.* If $\alpha \in R$, the following notation will be used: $C_k(\alpha) = \{x \in X : |L_k(x, \alpha)| = e\}$.

16. THEOREM. *There exists an $\alpha \in R$ such that for every $\beta \in R$, $C_k(\alpha) \subseteq C_k(\beta)$ for $k = 0, 1, \cdots, r$.*

Proof. Let m be the dimension of the convex set R. If $m = 0$, R is a single point, the best approximation is unique, and the theorem is true. If $m \geqq 1$ then R has a non-empty interior. Let α be an arbitrary point in the interior of the set R. It will be shown that this point satisfies the assertion of the theorem.

Let $\beta \neq \alpha$ be an arbitrary point of R. Extend the line $\overline{\alpha\beta}$ to the boundary of R, calling the points of intersection α_1 and β_1 as indicated by Figure 17. If β is a boundary point of R, then $\beta_1 = \beta$.



Now consider the functions $L_k[x, q\alpha_1 + (1 - q)\beta_1]$ where $0 < q < 1$. If $D^kP(x_0, \alpha_1) \neq D^kP(x_0, \beta_1)$ then one of the following relations holds:

(17) $$L_k(x_0, \alpha_1) < L_k[x_0, q\alpha_1 + (1 - q)\beta_1] < L_k(x_0, \beta_1),$$

(18) $$L_k(x_0, \beta_1) < L_k[x_0, q\alpha_1 + (1 - q)\beta_1] < L_k(x_0, \alpha_1).$$

Hence if $q \in (0, 1)$ is arbitrary then $x_0 \in C_k[q\alpha_1 + (1 - q)\beta_1]$ if and only if $x_0 \in C_k(\alpha_1)$, $x_0 \in C_k(\beta_1)$ and $D^kP(x_0, \alpha_1) = D^kP(x_0, \beta_1)$.

Since α is an interior point of the line segment $\overline{\alpha_1\beta_1}$ it follows from the above result that if $x_0 \in C_k(\alpha)$ then $x_0 \in C_k(\alpha_1)$, $x_0 \in C_k(\beta_1)$ and $D^kP(x_0, \alpha_1) = D^kP(x_0, \beta_1)$. This implies, using the result again, that $x_0 \in C_k[q\alpha_1 + (1 - q)\beta_1]$ for $0 \leqq q \leqq 1$, which completes the proof.

18. Corollary. *Suppose the best approximation is not unique, and let α be a point in the interior of R. Then if $P(x, \beta)$ is any best approximation, $D^k(x_0, \alpha) = D^k(x_0, \beta)$ for every extremal (x_0, k) of the approximation $P(x, \alpha)$ to f.*

Given an approximation $P(x, \alpha)$ to f we would like to have a means of telling if it is a best approximation. In addition, if $P(x, \alpha)$ is not a best approximation we would like to be able to find a better approximation. The following definition is useful in this endeavor.

19. *Definition.* Suppose $M[P(x, \alpha) - f] = d$. A polynomial $P(x, \beta)$ is said to satisfy Condition A with respect to the approximation $P(x, \alpha)$ to f, if for each extremal (x_0, k) of this approximation, $\operatorname{sgn} D^kP(x_0, \beta) = -\operatorname{sgn} D^k[P(x_0, \alpha) - f(x_0)]$.

20. Theorem. *A polynomial $P(x, \alpha)$ is a best approximation to f if and only if there is no polynomial $P(x, \beta)$ which satisfies Condition* A *with respect to the approximation.*

Proof. Suppose first that $P(x, \alpha)$ is not a best approximation, so that there exists a polynomial $P(x, \beta)$ such that $M[P(x, \beta) - f] < M[P(x, \alpha) - f]$. Then the polynomial $P(x, \beta - \alpha)$ satisfies Condition A with respect to the approximation $P(x, \alpha)$.

Next suppose that $P(x, \beta)$ satisfies Condition A with respect to the approximation $P(x, \alpha)$. Using the compactness of X and the continuity of the functions $D^kP(x, \beta)$ and $L_k(x, \alpha)$, one can establish that for each k there exists a constant $T_k > 0$ such that if $0 < t \leqq T_k$ then $\| L_k(x, \alpha + t\beta) \| < M[P(x, \alpha) - f]$. A proof of this result may be found in Remez [3, p. 38]. Hence if $0 < t \leqq \min [T_0, T_1, \cdots, T_r]$ then $M[P(x, \alpha + t\beta) - f] < M[P(x, \alpha) - f]$.

This theorem provides the basis for a computational scheme similar to the first method of Remez [3, p. 36]. In addition it provides a useful tool for investigating the uniqueness of a best approximation.

We know that if a differentiable function has a relative extremum at a point interior to its domain of definition, then the derivative must be zero at that point. Under proper assumptions this allows us to derive additional conditions which a best approximation must satisfy.

21. Theorem. *Let (x_0, k) be an extremal of the approximation $P(x, \alpha)$ to f, with $M[P(x, \alpha) - f] = d$. Suppose that at the point x_0 both $D^{k+1}[P(x, \alpha) - f]$ and $Dw_k(x)$ exist. If for every $\epsilon > 0$ there exist points x_1, x_2 in X such that $x_0 - \epsilon < x_1 < x_0 < x_2 < x_0 + \epsilon$ then $DL_k(x, \alpha) = 0$ at the point x_0.*

Proof. Suppose that $L_k(x_0, \alpha) = d$. Then

$$DL_k(x_0, \alpha) = \lim_{x \to x_0} \frac{L_k(x, \alpha) - d}{x - x_0}.$$

For all x, $L_k(x, \alpha) - d \leqq 0$. Hence if $x < x_0$ then $(L_k(x, \alpha) - d)/(x - x_0) \geqq 0$ while if $x > x_0$ then $(L_k(x, \alpha) - d)/(x - x_0) \leqq 0$. Since the approach to the limit may be made from either side of x_0, through points of X, it follows that the limit is zero. A similar argument holds if $L_k(x_0, \alpha) = -d$ so the proof is complete.

22. Corollary. *Suppose that $P(x, \alpha)$ is a best approximation to f, with α in the interior of R, and that the hypotheses of Theorem 21 are satisfied at the point (x_0, k). Then for any other best approximation $P(x, \beta)$ it follows that*

$$D^{k+1}P(x_0, \alpha) = D^{k+1}P(x_0, \beta).$$

Proof. From Corollary 18 we know that (x_0, k) is an extremal of the approximation $P(x, \beta)$ to f. Hence

$$DL_k(x_0, \alpha) = 0 = DL_k(x_0, \beta).$$

Using the product rule for differentiation we have

$$Dw_k(x_0)D^k[P(x_0, \alpha) - f(x_0)] + w_k(x_0)D^{k+1}[P(x_0, \alpha) - f(x_0)] = 0,$$

$$Dw_k(x_0)D^k[P(x_0, \beta) - f(x_0)] + w_k(x_0)D^{k+1}[P(x_0, \beta) - f(x_0)] = 0.$$

Since $D^k[P(x_0, \alpha) - f(x_0)] = D^k[P(x_0, \beta) - f(x_0)]$ and $w_k(x_0) \neq 0$ the result is established.

5. Uniqueness. The previous theorems give us considerable information about the extremals of a best approximation. We shall use these results to establish the following theorem.

Let $X = [-1, 1]$, $r = 1$, $\phi_i = x^i$, $i = 0, 1, \cdots, n$, and suppose $w_0'(x)$, $w_1'(x)$, and $f''(x)$ exist. The weight functions are assumed to be positive.

23. Theorem. *Under the above conditions one of the following assertions is true:*

(19) *The best approximation is unique.*

(20) *The best approximation is unique except for an additive constant; moreover, if P is any best approximation then DP is the unique best Chebyshev approximation of degree $n - 1$ to Df with weight function $w_1(x)$.*

We shall first discuss the conditions under which the second assertion will be true. Suppose there exists a best approximation $P(x, \gamma)$, with $M[P(x, \gamma) - f] = e$, such that $\| w_0(x)[P(x, \gamma) - f(x)] \| < e$. Then $\| w_1(x)D[P(x, \gamma) - f(x)] \| = e$. Now suppose that assertion (20) is false, so that there exists a polynomial $P(x, \alpha)$ such that $\| w_1(x)D[P(x, \alpha) - f(x)] \| < e$.

Define a polynomial

$$(21) \qquad P_q(x) \equiv q \int_0^x DP(x, \alpha)\, dx + (1 - q)P(x, \gamma), \quad \text{for} \quad 0 \leqq q \leqq 1.$$

Then

$$\| w_0(x)[P_q(x) - f(x)] \| \leqq q \left\| w_0(x) \left[\int_0^x DP(x, \alpha)\, dx - f(x) \right] \right\| + (1 - q) \| w_0(x)[P(x, \gamma) - f(x)] \|.$$

Hence there exists a $t > 0$ such that if $0 < q \leqq t$ then $\| w_0(x)[P_q(x) - f(x)] \| < e$. However, $\| w_1(x)D[P_q(x) - f(x)] \| \leqq q \| w_1(x)D[P(x, \alpha) - f(x) \| + (1 - q) \| w_1(x)D[P(x, \gamma) - f] \| < e$. Hence if $q < t$ then $P_q(x)$ is a better approximation to f than is $P(x, \gamma)$; this is a contradiction. Therefore, if there exists any best approximation $P(x, \gamma)$ such that $\| L_0(x, \gamma) \| < e$ then (20) is true.

From the theory of equations we know that if two polynomials of degree n agree on a set of $s + t$ points, where $s > 0$ and $t \geqq 0$, if their first derivatives agree on a subset of t of these points and if $s + 2t \geqq n + 1$, then the polynomials are identical. In addition if $s + t$ distinct points are specified, with $s > 0$, then it is possible to find a polynomial of degree $\leqq n$ which has arbitrary values on the $s + t$ points, and whose derivative has arbitrary values on a subset of t of these points, provided $s + 2t \leqq n + 1$. For the remainder of the proof we shall assume, for each $\alpha \in R$, that $\| L_0(x, \alpha) \| = e$. Let β be a point in the interior of R. For notational convenience define:

$$S \equiv C_0(\beta) \tag{22}$$

$$T \equiv C_1(\beta) \tag{23}$$

$$U = S \cap T \cap (-1, 1). \tag{24}$$

Let s, t, u be the cardinality of S, T, U respectively.

Using Corollary 18 and Corollary 22 it follows that if $P(x, \alpha)$ is any best approximation then

$$P(x, \beta) = P(x, \beta) \ \forall\, x \in S, \tag{25}$$

$$DP(x, \alpha) = DP(x, \beta) \ \forall\, x \in S \cap (-1, 1), \tag{26}$$

$$DP(x, \alpha) = DP(x, \beta) \ \forall\, x \in T, \tag{27}$$

$$D^2P(x, \alpha) = D^2P(x, \beta) \ \forall\, x \in T \cap (-1, 1). \tag{28}$$

If we assume that α and β are distinct we can now give a lower bound for n, the degree of the approximating polynomials, in terms of s, t, u. We can compute an upper bound for the degree of a polynomial which satisfies Condition A with respect to the approximation $P(x, \beta)$, in terms of the same s, t, u. We shall show that the former is $\geqq$ the latter, leading to a contradiction of the assumption $\alpha \neq \beta$, and completing the proof.

Since the points $+1$ and -1 can be in S and T independently, it is necessary to consider each possible placement of these points as a separate case. The arguments involved in each case are similar, so that only one example will be given. A summary of all cases is given in Table 24.

Consider the case of $-1 \in S$, $+1 \in S$, $-1 \in T$, $+1 \in T$. Then

$$DP(x, \alpha) = DP(x, \beta) \quad \text{for the} \quad s - 2 \quad \text{points} \quad x \in S \cap (-1, 1), \tag{29}$$

$$DP(x, \alpha) = DP(x, \beta) \quad \text{for the} \quad t \quad \text{points} \quad x \in T, \tag{30}$$

$$D^2P(x, \alpha) = D^2P(x, \beta) \quad \text{for the} \quad t - 2 \quad \text{points} \quad x \in T \cap (-1, 1). \tag{31}$$

Since $U = S \cap T \cap (-1, 1)$, u of the relations (29) are identical to those given in (30). Hence the polynomials $DP(x, \alpha)$ and $DP(x, \beta)$, which are of degree $n - 1$, agree on a set of $(s - 2 - u) + t$ points, and *their* derivatives agree on a subset of $t - 2$ of these points. Hence if $n - 1 \leqq (s - 2 - u) + t + (t - 2) - 1$ then $DP(x, \alpha) \equiv DP(x, \beta)$ and $\alpha = \beta$. That is, the assumption $\alpha \neq \beta$ forces the conclusion $n \geqq s + 2t - u - 3$.

A polynomial $P(x, \gamma)$ will satisfy Condition A with respect to $P(x, \beta)$ if

$$\operatorname{sgn} P(x, \gamma) = -\operatorname{sgn} L_0(x, \beta) \qquad \text{for} \quad x \in S,$$

$$\operatorname{sgn} DP(x, \gamma) = -\operatorname{sgn} L_1(x, \beta) \qquad \text{for} \quad x \in T.$$

In the case we are considering there are s points in S, t points in T and $t - u - 2$ points which are in T but not in S. Thus a polynomial which is to have arbitrary values on $S \cup T$, and whose derivative is to have arbitrary values on T, must satisfy a total of $s + (t - u - 2) + t$ conditions. It follows that there is a polynomial $P(x, \gamma)$ of degree $\leqq s + 2t - u - 3$ which satisfies Condition A. This completes the proof in this case.

24. TABLE.

Case	$\alpha = \beta$ unless $n \geqq$	Condition A is satisfied by a polynomial of degree
$-1 \in S, +1 \in S, -1 \in T, +1 \in T$	$s + 2t - u - 3$	$s + 2t - u - 3$
$-1 \in S, -1 \in T, +1 \in T$	$s + 2t - u - 2$	$s + 2t - u - 2$
$+1 \in S, -1 \in T, +1 \in T$	$s + 2t - u - 2$	$s + 2t - u - 2$
$-1 \in T, +1 \in T$	$s + 2t - u - 1$	$s + 2t - u - 1$
$-1 \in S, +1 \in S, -1 \in T$	$s + 2t - u - 2$	$s + 2t - u - 2$
$-1 \in S, -1 \in T$	$s + 2t - u - 1$	$s + 2t - u - 2$
$+1 \in S, -1 \in T$	$s + 2t - u - 1$	$s + 2t - u - 1$
$-1 \in T$	$s + 2t - u$	$s + 2t - u - 1$
$-1 \in S, +1 \in S, +1 \in T$	$s + 2t - u - 2$	$s + 2t - u - 2$
$-1 \in S, +1 \in T$	$s + 2t - u - 1$	$s + 2t - u - 1$
$+1 \in S, +1 \in T$	$s + 2t - u - 1$	$s + 2t - u - 2$
$+1 \in T$	$s + 2t - u$	$s + 2t - u - 1$
$-1 \in S, +1 \in S$	$s + 2t - u - 1$	$s + 2t - u - 1$
$-1 \in S$	$s + 2t - u$	$s + 2t - u - 1$
$+1 \in S,$	$s + 2t - u$	$s + 2t - u - 1$
No points of ± 1 in S or T	$s + 2t - u + 1$	$s + 2t - u - 1$

5. Remarks. The theorem just proved is, of course, true if the interval $[-1, 1]$ is replaced by any finite interval. A valuable application of this theorem is the case of $w_0(x) \equiv 1$ and $w_1(x) \equiv k$, a positive constant. In this case we can readily determine which of (19), (20) characterizes the solution to the approximation problem. Let $P(x)$ be such that $DP(x)$ is the unique best Chebyshev approximation of degree $n - 1$ to $Df(x)$. Pick the constant coefficient of P so that $\| P - f \|$ is as small as possible. Then if $\| P - f \| \geqq k \| D(P - f) \|$ it follows that the best approximation will be unique; otherwise, (20) characterizes the solution. The computational aspects of this problem are quite interesting. A computational scheme will be presented in a later paper.

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