The Evaluation of Some Definite Integrals Involving Bessel Functions Which Occur in Hydrodynamics and Elasticity

By A. H. Van Tuyl

1. Introduction. Integrals of the form

$$I_1(m, \nu) = \int_0^\infty e^{-xt} J_\nu(yt) J_\nu(bt) t^m \, dt \tag{1.1}$$

and

$$I_2(m, \nu) = \int_0^\infty e^{-xt} J_{\nu-1}(yt) J_\nu(bt) t^{m+1} \, dt \tag{1.2}$$

occur in various problems of potential theory, hydrodynamics, and elasticity (A. Weinstein [1, 2], H. Bateman [3, p. 417], and A. G. Webster [4, p. 367–375]). In [1]–[4], ν is an integer, x, y, and b are real, and m is a positive or negative integer, or zero, such that (1.1) and (1.2) converge. When ν is an integer, special cases of (1.1) and (1.2) have often been expressed in the literature in terms of elliptic integrals, the modulus of which is an elementary algebraic function of x, y, and b. The object of the present paper is to obtain closed expressions for $I_1(m, \nu)$ in terms of known functions, assuming that ν is complex when $m \geqq 0$, and an integer when $m < 0$, and to find recurrence relations involving $I_1(m, \nu)$ and $I_2(m, \nu)$. It is assumed throughout that x, y, and b are complex. When $m \geqq 0$, $I_1(m, \nu)$ is evaluated in closed form in terms of associated Legendre functions of the second kind of the form $Q^r_{\nu-1/2}(z)$; when $m < 0$ and ν is an integer, it is expressed in terms of associated Legendre functions of the form $P^{-n}_{n-m}(z)$, and Appell's generalized hypergeometric function F_1. By means of either the recurrence relations in Section 4, or certain differentiation formulas, explicit expressions of the same form as in Sections 2 and 3 can also be obtained for $I_2(m, \nu)$. When ν is an integer, these explicit expressions for $I_1(m, \nu)$ and $I_2(m, \nu)$ can be obtained in terms of standard elliptic integrals by means of well-known reduction formulas.

The more general integral

$$I(\mu, \nu; \lambda) = \int_0^\infty e^{-ct} J_\mu(at) J_\nu(bt) t^\lambda \, dt \tag{1.3}$$

has been treated by G. Eason, B. Noble, and I. N. Sneddon [5] when μ, ν, and λ are integers such that (1.3) converges, and a, b, and c are real. A summary of [5] is given by Y. L. Luke [6, p. 314]. The results of [5] can be used for the calculation of (1.3) in terms of elliptic integrals for all integers μ, ν, and λ, $\lambda \leqq 1$, for which it converges. However, explicit expressions are given in [5] only when (1.3) is of the form of certain special cases of (1.1) and (1.2). The calculation of (1.3) for other values of μ, ν, and λ can be carried out by use of recurrence relations obtained in [5]. The three recurrence relations obtained in Section 4 of the present paper are seen not to be special cases of the recurrence relations in [5].

Received August 22, 1963. Revised January 31, 1964.

When ν is an integer, and either $m \geqq 0$ or $b = y$, (1.1) and (1.2) can be expressed in terms of complete elliptic integrals of the first and second kinds. When $m < 0$ and $b = y$, complete elliptic integrals of the third kind occur also. Examples of the latter can be obtained from results of several authors (W. M. Hicks [7, p. 628], G. M. Minchin [8, p. 354], A. Van Tuyl [9], M. R. Shura-Bura [10], M. A. Sadowsky and E. Sternberg [11], G. E. Pringle [12, p. 385 and 392], and H. E. Fettis [13]). In [7], [8], and [11], integrals of the form of (1.1) or (1.2) are not considered explicitly. A quantity known to be proportional to $I_2(-1, 1)$ is evaluated in terms of elliptic integrals in [7] and [8], and quantities proportional to $I_1(-1, 1)$, $I_2(-1, 1)$, and $I_2(-2, 1)$ are evaluated in [11]. The evaluation of $I_2(-1, 1)$ and $I_2(-2, 1)$ by numerical integration has been discussed by G. P. Weeg [14].

For convergence of (1.1) and (1.2) at the lower limit, it is necessary to have $\operatorname{Re}(2\nu + m + 1) > 0$, and for convergence at the upper limit, all four quantities $\operatorname{Re}(x \pm ib \pm iy)$ must be either positive or zero. In the latter case, we must also have either $m < 1$ or $m < 0$, depending on the values of x, b, and y. When $\operatorname{Re}(2\nu + m + 1) > 0$ and $\operatorname{Re}(x \pm ib \pm iy) > 0$, it is easily shown that (1.1) and (1.2) are analytic functions of each of the variables x, y, and b, and that differentiations of (1.1) and (1.2) with respect to x, y, and b can be carried out under the integral sign.

Defining

$$\phi(x, y) = y^{1-\nu} I_2(m - 1, \nu), \tag{1.4}$$

$$\psi(x, y) = -y^{\nu} I_1(m, \nu), \tag{1.5}$$

it is seen that $\phi(x, y)$ and $\psi(x, y)$ satisfy the differential equations

$$y^{2\nu-1} \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \tag{1.6}$$

$$y^{2\nu-1} \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{1.7}$$

Despite the fact that ν may be complex, it is convenient, following A. Weinstein [15], to call $\phi(x, y)$ a generalized axially symmetric potential in $2\nu + 1$ dimensions, and $\psi(x, y)$ its associated stream function. The analogy with the case when ν is an integer has proven fruitful. Similarly, writing

$$\phi(x, b) = b^{-\nu} I_1(m, \nu), \tag{1.8}$$

$$\psi(x, b) = -b^{\nu+1} I_2(m, \nu + 1), \tag{1.9}$$

we have

$$b^{2\nu+1} \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \tag{1.10}$$

$$b^{2\nu+1} \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{1.11}$$

In [1]–[4], integrals $I_1(m, n)$ and $I_2(m, n)$ arise from the solution of special cases of these equations.

We note that $I_2(m, \nu)$ can be calculated in terms of $I_1(m, \nu)$ by means of the relations

$$I_2(m, \nu) = y^{-\nu} \frac{\partial}{\partial y} [y^\nu I_1(m, \nu)] \tag{1.12}$$

$$= -b^{\nu-1} \frac{\partial}{\partial b} [b^{1-\nu} I_1(m, \nu - 1)]. \tag{1.13}$$

Thus, explicit expressions for $I_2(m, \nu)$, $m \geqq 0$, and $I_2(m, n)$, $m < 0$, can be obtained from the results of Sections 2 and 3 respectively. Alternatively, we can calculate $I_2(m, \nu)$ in terms of $I_1(m, \nu)$ by means of one of the recurrence relations in Section 4.

2. Evaluation of $I_1(m, \nu)$ for $m \geqq 0$. From Watson [16, p. 389], equation (1), we have

$$\begin{aligned} I_1(m, \nu) &= \frac{(by)^\nu \Gamma(2\nu + m + 1)}{\pi x^{2\nu+m+1} \Gamma(2\nu + 1)} \\ &\quad \cdot \int_0^\pi F\left(\frac{2\nu + m + 1}{2}, \frac{2\nu + m + 2}{2}; \nu + 1; -\frac{p^2}{x^2}\right) \sin^{2\nu} \phi \, d\phi, \end{aligned} \tag{2.1}$$

where $F(a, b; c; z)$ is the hypergeometric function, and $p^2 = b^2 + y^2 - 2by \cos \phi$. Equation (2.1) holds for the same values of the parameters for which (1.1) and (1.2) converge. From [6, p. 75], it follows that it is sufficient to assume that $|\arg b| \leqq \pi/2$ and $|\arg y| \leqq \pi/2$. It is assumed in (2.1) that $|\arg(1 + p^2/x^2)| < \pi$, and that $\arg(\sin^2 \phi) = 0$.

Setting $b = c + n + \epsilon$ in [17, p. 109], equation (3), where $n \geqq 0$ is an integer, and where c and $a - c - n$ are not negative integers, and letting $\epsilon \to 0$, we obtain

$$\begin{aligned} F(a, c + n; c; z) &= \frac{\Gamma(c)\Gamma(c - a + n)}{\Gamma(c + n)\Gamma(c - a)} (1 - z)^{-a} \\ &\quad \cdot F\left(a, -n; a - c - n + 1; \frac{1}{1 - z}\right) \end{aligned} \tag{2.2}$$

when $|\arg(1 - z)| < \pi$. Hence, noting that the preceding conditions are satisfied in each case, we have

$$F\left(\frac{2\nu + m + 1}{2}, \frac{2\nu + m + 2}{2}; \nu + 1; z\right)$$

$$\begin{aligned} &= \frac{\Gamma(\nu + 1)\Gamma(\frac{1}{2})}{\Gamma(\nu + r + 1)\Gamma(\frac{1}{2} - r)} (1 - z)^{-\nu-r-1/2} \\ &\quad \cdot F\left(\nu + r + \tfrac{1}{2}, -r; \tfrac{1}{2}; \frac{1}{1 - z}\right), \quad m = 2r, \end{aligned} \tag{2.3}$$

$$\begin{aligned} &= \frac{\Gamma(\nu + 1)\Gamma(-\frac{1}{2})}{\Gamma(\nu + r + 1)\Gamma(-\frac{1}{2} - r)} (1 - z)^{-\nu-r-3/2} \\ &\quad \cdot F\left(\nu + r + \frac{3}{2}, -r; \frac{3}{2}; \frac{1}{1 - z}\right), \quad m = 2r + 1, \end{aligned} \tag{2.4}$$

$r = 0, 1, \cdots$, $|\arg(1 - z)| < \pi$. From the preceding, using Legendre's duplication formula for $\Gamma(z)$, we obtain

$$
\begin{aligned}
I_1(2r, \nu) = &\ \frac{(-1)^r\, 2^{r-\nu-1/2}\Gamma(r + \frac{1}{2})}{\pi(by)^{r+1/2}\Gamma(\nu + \frac{1}{2})} \\
&\cdot \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\Gamma(\nu + r + \frac{1}{2} + i)}{\Gamma(\frac{1}{2} + i)} \left(\frac{x^2}{2by}\right) I(r + i, \nu)
\end{aligned} \tag{2.5}
$$

and

$$
\begin{aligned}
I_1(2r + 1, \nu) = &\ \frac{(-1)^r\, 2^{r-\nu-1/2}\Gamma(r + \frac{3}{2})x}{\pi(by)^{r+3/2}\Gamma(\nu + \frac{1}{2})} \\
&\cdot \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\Gamma(\nu + r + \frac{3}{2} + i)}{\Gamma(\frac{3}{2} + i)} \left(\frac{x^2}{2by}\right)^i I(r + 1 + i, \nu),
\end{aligned} \tag{2.6}
$$

$r = 0, 1, \cdots$, where

$$
I(m, \nu) = \int_0^{\pi} \frac{\sin^{2\nu}\phi\, d\phi}{(\beta - \cos\phi)^{m+\nu+1/2}}, \tag{2.7}
$$

with

$$
\beta = \frac{x^2 + b^2 + y^2}{2by}. \tag{2.8}
$$

We see that (2.7) is an analytic function of β when the β-plane is cut along the real axis from 1 to $-\infty$. In order for (2.5) and (2.6) to be real when all parameters are real and positive, we must then have $|\arg(\beta - \cos\phi)| < \pi$ in (2.7). The cut β-plane is more extensive than the region in which (2.1) converges, hence, (2.5) and (2.6) give an analytic continuation of (2.1).

Substituting $\cos\phi = t$ in (2.7) and comparing with Hobson [18, p. 195], equation (20), we obtain

$$
I(m, \nu) = (-1)^m\, 2^{\nu+1/2} \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu + m + \frac{1}{2})} (\beta^2 - 1)^{-m/2} Q^m_{\nu-1/2}(\beta) \tag{2.9}
$$

throughout the cut β-plane, where $Q_a^{\ b}(z)$ is the associated Legendre function of the second kind, and where $|\arg(\beta - 1)| < \pi$, $|\arg(\beta + 1)| < \pi$. We note that the conditions $\arg(\sin^2\phi) = 0$ and $|\arg(\beta - \cos\phi)| < \pi$ which hold in (2.7) are required in the preceding reference. Finally, from (2.5), (2.6), and (2.9), we have

$$
I_1(2r, \nu) = \frac{2^r\Gamma(r + \frac{1}{2})}{\pi(by)^{r+1/2}} \sum_{i=0}^{r} \binom{r}{i} \frac{1}{\Gamma(\frac{1}{2} + 1)} \left(\frac{x^2}{2by}\right)^i (\beta^2 - 1)^{-(r+i)/2} Q^{r+i}_{\nu-1/2}(\beta) \tag{2.10}
$$

and

$$
\begin{aligned}
I_1(2r + 1, \nu) = &\ - \frac{2^r\Gamma(r + \frac{3}{2})}{\pi(by)^{r+3/2}}\, x \\
&\cdot \sum_{i=0}^{r} \frac{1}{\Gamma(\frac{3}{2} + i)} \left(\frac{x^2}{2by}\right)^i (\beta^2 - 1)^{-(r+i+1)/2} Q^{r+i+1}_{\nu-1/2}(\beta),
\end{aligned} \tag{2.11}
$$

where β satisfies the preceding conditions, and $r = 0, 1, \cdots$.

The result

$$(2.12)\qquad I_1(0,\nu) = \frac{(by)^\nu}{\pi}\int_0^\pi \frac{\sin^2\phi^\nu\,d\phi}{(x^2+b^2+y^2-2by\cos\phi)^{\nu+1/2}}$$

was obtained by A. Weinstein ([15], equations (9) and (12)), and the expression

$$(2.13)\qquad I_1(0,\nu) = \frac{1}{\pi\sqrt{by}}\,Q_{\nu-1/2}(\beta)$$

was obtained by Watson ([16, p. 389], equation (2)).

When ν is an integer n, $n \geqq 0$, we can express $I(m,\nu)$, and hence, $Q^m_{\nu-1/2}(\beta)$, in terms of complete elliptic integrals of the first and second kinds. Substituting $\cos\phi = 2sn^2u - 1$ in (2.7) when $\nu = n$, where the modulus, k, of $sn\,u$ is given by

$$(2.14)\qquad k^2 = \frac{2}{1+\beta} = \frac{4by}{x^2+(b+y)^2},$$

we obtain

$$(2.15)\qquad I(m,n) = 2^{n-m+1/2}k^{2n+2m+1}\int_0^K \frac{sn^{2n}u\,cn^{2n}u\,du}{dn^{2m+2n}u}.$$

From (2.14), it follows that the cut in the β-plane from 1 to $-\infty$ corresponds to cuts in the k^2-plane from 1 to ∞ and from 0 to $-\infty$ along the real axis. Concerning the path of integration in (2.15), it is known that the residues of the integrand are zero at all of its poles. Hence, it follows that (2.15) is independent of the path of integration joining the points 0 and K. In particular, we can always take the latter to be the straight line between 0 and K.

The right side of (2.15) can be evaluated in terms of elliptic integrals by means of well-known reduction formulas (P. F. Byrd and M. D. Friedman [19, p. 191–198]). Alternatively, we can compute $Q^m_{n-1/2}(\beta)$ for $m \geqq 0$ and $n \geqq 0$ by use of the recurrence relations for $Q_\nu(z)$ ([18, p. 290], equations (164) and (166)), starting from the values of $Q_{-1/2}(\beta)$, $Q_{1/2}(\beta)$, $Q^1_{-1/2}(\beta)$, and $Q^1_{1/2}(\beta)$. From (2.9) and (2.15), referring to [19], we have

$$(2.16)\qquad Q_{-1/2}(\beta) = kK,$$

$$(2.17)\qquad Q_{1/2}(\beta) = \frac{2}{k}(K-E) - kK,$$

$$(2.18)\qquad Q^1_{-1/2}(\beta) = -\frac{kk'E}{2},$$

and

$$(2.19)\qquad Q^1_{1/2}(\beta) = \frac{1}{2kk'^3}[2k'^2K - (2-k^2)E],$$

where K and E are the complete elliptic integrals of the first and second kinds, respectively, and $k'^2 = 1 - k^2$. Both k and k' are single valued when the k^2-plane is cut from 0 to $-\infty$ and from 1 to ∞. Another method for the computation of $Q^m_{n-1/2}(\beta)$ in terms of elliptic integrals can be obtained from [7], Sections II and III.

Finally, we note that $I_1(0,n)$ and $I_1(1,n)$ satisfy simple recurrence relations.

From (2.11), we have

$$I_1(1, n) = -\frac{x}{\pi(by)^{3/2}}(\beta^2 - 1)^{-1/2}Q^1_{n-1/2}(\beta). \tag{2.20}$$

From (2.13), (2.22), and (2.16), we obtain the recurrence relations

$$(n + \tfrac{1}{2})I_1(0, n + 1) - 2n\beta I_1(0, n) + (n - \tfrac{1}{2})I_1(0, n - 1) = 0 \tag{2.21}$$

and

$$(n - \tfrac{1}{2})I_1(1, n + 1) - 2n\beta I_1(1, n) + (n + \tfrac{1}{2})I_1(1, n - 1) = 0. \tag{2.22}$$

Expressions for $I_1(0, 0)$, $I_1(0, 1)$, $I_1(1, 0)$, and $I_1(1, 1)$ in terms of elliptic integrals ollow from (2.13), (2.20), and equations (2.16) through (2.19).

3. Evaluation of $I_1(m, n)$ for $m < 0$. From [17, p. 105 and 107], equations (1), (10), (18), and (37), we have the relation

$$\begin{aligned}(-z)^{b-c}&(1 - z)^{c-a-b}F(1 - b, c - b; a + 1 - b; 1/z)\\ &= \frac{\Gamma(1 - c)\Gamma(a + 1 - b)}{\Gamma(1 - b)\Gamma(a + 1 - c)}F(a, b; c; z) - \frac{\Gamma(c)\Gamma(1 - c)\Gamma(a + 1 - b)}{\Gamma(2 - c)\Gamma(c - b)\Gamma(a)}\\ &\qquad \cdot e^{i\pi(c-1)}z^{1-c}(1 - z)^{c-a-b}F(1 - a, 1 - b; 2 - c; z)\end{aligned} \tag{3.1}$$

when $\operatorname{Im} z > 0$. Substituting $a = r - n + \frac{1}{2}$, $b = r - n$, and $c = 1 - n + \epsilon$, $n \geqq r \geqq 1$, $\epsilon > 0$, and letting $\epsilon \to 0$, we obtain

$$\begin{aligned}F&\left(n - r + \frac{1}{2}, n - r + 1; n + 1; z\right)\\ &= \frac{\Gamma(n + 1)\Gamma(r - n + \frac{1}{2})}{\Gamma(r)\Gamma(\frac{3}{2})}z^{r-n-1}F\left(n - r + 1, 1 - r; \frac{3}{2}; 1/z\right)\\ &\quad + (-1)^r\frac{\Gamma(n + 1)\Gamma(r - n + \frac{1}{2})\Gamma(n)}{\Gamma(n - r + 1)\Gamma(r)\Gamma(r + \frac{1}{2})}\\ &\qquad \cdot z^{-n}(1 - z)^{2r-n-1/2}F\left(r - n + \frac{1}{2}, r - n; 1 - n; z\right),\end{aligned} \tag{3.2}$$

$\operatorname{Im} z > 0$. The second hypergeometric function on the right side of (3.2) is of the form $F(a, -m; -m - l; z)$, where $m \geqq 0$, $l \geqq 0$, and is defined as in [15, p. 101], equation (3). We see that it remains a polynomial throughout the limiting process $\epsilon \to 0$. Noting that both sides of (3.2) are analytic in the z-plane cut along the real axis from 1 to ∞, it follows by analytic continuation that (3.2) is valid for $|\arg(1 - z)| < \pi$. Similarly, setting $a = r - n - \frac{1}{2}$, $b = r - n$, and $c = 1 - n + \epsilon$, $n \geqq r \geqq 1$, $\epsilon > 0$, and letting $\epsilon \to 0$, we have

$$\begin{aligned}F&\left(n - r + 1, n + r + \frac{3}{2}; n + 1; z\right)\\ &= \frac{\Gamma(n + 1)\Gamma(r - n - \frac{1}{2})}{\Gamma(r)\Gamma(\frac{1}{2})}z^{r-n-1}F\left(n - r + 1, 1 - r; \frac{1}{2}; 1/z\right)\\ &\quad + (-1)^r\frac{\Gamma(n + 1)\Gamma(r - n - \frac{1}{2})\Gamma(n)}{\Gamma(n - r + 1)\Gamma(r)\Gamma(r - \frac{1}{2})}\\ &\qquad \cdot z^{-n}(1 - z)^{2r-n-3/2}F\left(r - n - \frac{1}{2}, r - n; 1 - n; z\right)\end{aligned} \tag{3.3}$$

when $|\arg(1 - z)| < \pi$. Finally, from (3.2), (3.3), and (2.1), using Legendre's duplication formula for $\Gamma(z)$ together with the functional equations, we find that

$$I_1(-2r, n) = \frac{2^r n!}{\pi(2n)!}(by)^{r-1}\left\{-2^{n-2r}x\sum_{i=0}^{r-1}\frac{(n - r + i)!}{(n - 1 - i)!(2i + 1)!}\right.$$

$$\tag{3.4} \cdot\left(\frac{2x^2}{by}\right)^i J(n - r + i + 1, n) + 2^{2r-n-2}\frac{(2n - 2r)!}{(2r - 1)!}(2by)^{1/2}$$

$$\left.\cdot\sum_{i=0}^{n-r}\frac{(r - 1 + i)!}{(n - r - i)!(2i)!}\left(\frac{2x^2}{by}\right)^i K(2r - n, r + i, n)\right\}$$

and

$$I_1(1 - 2r, n) = \frac{2^r n!}{\pi(2n)!}(by)^{r-1}\left\{2^{n-2r}\sum_{i=0}^{r-1}\frac{(n - r + i)!}{(r - 1 - i)!(2i)!}\right.$$

$$\tag{3.5} \cdot\left(\frac{2x^2}{by}\right)^i J(n - r + i + 1, n) - 2^{2r-n-2}\frac{(2n - 2r + 1)!}{(2r - 2)!}(2by)^{-1/2}x$$

$$\left.\cdot\sum_{i=0}^{n-r}\frac{(r - 1 + i)!}{(n - r - i)!(2i + 1)!}\left(\frac{2x^2}{by}\right)^i K(2r - n - 1, r + i, n)\right\},$$

$r = 1, 2, \cdots$, where

$$\tag{3.6} J(m, n) = \int_0^\pi \frac{\sin^{2n}\phi\, d\phi}{(\alpha - \cos\phi)^m}$$

and

$$\tag{3.7} K(q, m, n) = \int_0^\pi \frac{(\beta - \cos\phi)^{q-1/2}\sin^{2n}\phi\, d\phi}{(\alpha - \cos\phi)^m},$$

with

$$\tag{3.8} \alpha = \frac{b^2 + y^2}{2by},$$

and with β defined as in (2.8). It follows from (3.4) and (3.5) that q may be a positive or negative integer, or zero, and that $m \leqq n$ in (3.6) and (3.7). We see that (3.6) and (3.7) are analytic functions of α when the α-plane is cut along the real axis from -1 to 1, and that (3.7) is an analytic function of β when the β-plane is cut from 1 to $-\infty$. In order for $K(q, m, n)$ to be positive when $\alpha > 1$ and $\beta > 1$, we choose $|\arg(\beta - \cos\phi)| < \pi$ in (3.7).

Substituting $\cos\phi = t$ in (3.6) and comparing with [18, p. 195], equation (20), we find that

$$\tag{3.9} J(m, n) = \frac{2^{n+1/2}\Gamma(n + \frac{1}{2})}{(m)} e^{(n-m+1/2)\pi i}(\alpha^2 - 1)^{(n-m)/2+1/4}Q_{n-1/2}^{m-n-1/2}(\alpha)$$

when the α-plane is cut from 1 to $-\infty$ along the real axis, with $|\arg(\alpha - 1)| < \pi$ and $|\arg(\alpha + 1)| < \pi$. Using Whipple's relation in (3.9) [18, p. 245], equation (92), we obtain

$$\tag{3.10} J(m, n) = \frac{(2n)!\,\pi}{2^n n!}(\alpha^2 - 1)^{(n-m)/2}P_{n-m}^{-n}\left(\frac{\alpha}{\sqrt{\alpha^2 - 1}}\right)$$

in the cut α-plane when Re $\alpha > 0$. With $|\arg(\alpha - 1)| < \pi$ and $|\arg(\alpha+1)| < \pi$ as before, we see that Re $(\alpha/\sqrt{\alpha^2 - 1}) > 0$ both for Re $\alpha < 0$ and Re $\alpha > 0$, and that the imaginary axis is mapped twice onto the segment $0 \leqq \alpha/\sqrt{\alpha^2 - 1} \leqq 1$. Writing

$$(3.11) \qquad J(m, n) = \frac{(2n)!\,\pi}{2^n n!}\bar{J}(m, n),$$

we can verify that

$$(3.12) \quad \bar{J}(m,n) = (\alpha^2 - 1)^{(n-m)/2} P_{n-m}^{-n}\left(\frac{\alpha}{\sqrt{\alpha^2 - 1}}\right), \qquad \text{Re}\,\alpha > 0,$$

$$(3.13) \qquad = e^{\mp i\pi m/2}(\alpha_1^2 + 1)^{(n-m)/2} P_{n-m}^{-n}\left(\frac{\alpha_1}{\sqrt{\alpha_1^2 + 1}}\right), \quad \alpha = \pm i\alpha_1, \alpha_1 > 0,$$

$$(3.14) \qquad = (-1)^n(\alpha^2 - 1)^{(n-m)/2} P_{n-m}^{-n}\left(\frac{\alpha}{\sqrt{\alpha^2 - 1}}\right), \qquad \text{Re}\,\alpha < 0$$

when the α-plane is cut along the real axis from -1 to 1, where $P_{n-m}^{-n}(x)$ is defined in the usual way when $0 < x < 1$ [18, p. 99]. From (3.10) and [18, p. 100], equation (28), we find that

$$(3.15) \quad \bar{J}(m,n) = \frac{(\alpha^2 - 1)^{(n-m)/2}}{(\alpha + \sqrt{\alpha^2 - 1})^n} \sum_{i=0}^{n-m} \frac{(n - m + i)!}{(n - m - i)!(n + i)!\,i!\,2^i} \cdot \frac{(\alpha^2 - 1)^{-i/2}}{(\alpha + \sqrt{\alpha^2 - 1})^i}$$

in the cut plane when Re $\alpha > 0$, and with $n \geqq m$ and $n \geqq 0$. Noting that both sides of (3.15) are analytic when the α-plane is cut along the real axis from -1 to 1, it follows by analytic continuation that (3.15) holds throughout the cut α-plane. It can be verified that (3.9) through (3.15) remain valid when $m < 0$, and that (3.15) simplifies to 2^{-n} when $m = 0$.

Similarly, substituting $\cos\phi = 2t - 1$ in (3.7) and comparing with [17, p. 231], equation (5), we have

$$(3.16) \quad \begin{aligned} K(q, m, n) = {} & \frac{(2n)!\,\pi}{2^{2n}(n!)^2} \cdot \frac{(\beta + 1)^{q-1/2}}{(\alpha + 1)^m} \\ & \cdot F_1\left(n + \frac{1}{2}, m, \frac{1}{2} - q, 2n + 1; \frac{2}{\alpha + 1}, \frac{2}{\beta + 1}\right), \end{aligned}$$

where $F_1(a, b, b', c; x, y)$ is the first of Appell's generalized hypergeometric functions of two variables. From (3.4), (3.5), (3.11) and (3.16), we find that

$$(3.17) \quad \begin{aligned} I_1(-2r, n) = {} & -2^{-r}(by)^{r-1}x \\ & \cdot \sum_{i=0}^{r-1} \frac{(n - r + i)!}{(n - 1 - i)!\,(2i + 1)!}\left(\frac{2x^2}{by}\right)^i \bar{J}(n - r + i + 1, n) \\ & + \frac{2^{3r-3n-3/2}(2n - 2r)!}{(2r - 1)!\,n!}(by)^{r-1/2} \sum_{i=0}^{n-r} \frac{(r - 1 + i)!}{(n - r - i)!\,(2i)!}\left(\frac{2x^2}{by}\right)^i \\ & \cdot \frac{(\beta + 1)^{2r-n-1/2}}{(\alpha + 1)^{r+i}} F_1\left(n + \frac{1}{2}, r + i, n - 2r + \frac{1}{2}, 2n + 1;\right. \\ & \left. \frac{2}{\alpha + 1}, \frac{2}{\beta + 1}\right); \end{aligned}$$

$$
\begin{aligned}
&I_1(1-2r,n) \\
&= 2^{-r}(by)^{r-1}\sum_{i=0}^{r-1}\frac{(n-r+i)!}{(r-1-i)!\,(2i)!}\left(\frac{2x^2}{by}\right)^i \bar{J}(n-r+i+1,n) \\
&\quad - \frac{2^{3r-3n-5/2}(2n-2r+1)!}{(2r-2)!\,n!}(by)^{r-3/2}x\sum_{i=0}^{n-r}\frac{(r-1+i)!}{(n-r-i)!\,(2i+1)!}\left(\frac{2x^2}{by}\right)^i \\
&\quad \cdot \frac{(\beta+1)^{2r-n-3/2}}{(\alpha+1)^{r+i}}F_1\left(n+\frac{1}{2}, r+i, n-2r+\frac{3}{2}, 2n+1;\right. \\
&\qquad\qquad\left.\frac{2}{\alpha+1}, \frac{2}{\beta+1}\right), r \geqq 1.
\end{aligned} \tag{3.18}
$$

Finally, we consider the evaluation of $K(q, m, n)$ in terms of elliptic integrals. Let k^2 be given by (2.8), and let

$$
l^2 = \frac{2}{\alpha+1} = \frac{4by}{(b+y)^2}. \tag{3.19}
$$

We see that the cut in the α-plane from -1 to 1 corresponds to a cut in the l^2-plane along the real axis from 1 to ∞. Substituting $\cos\phi = 2sn^2u - 1$ in (3.7), we have

$$
K(q,m,n) = \frac{2^{2n-m+q+1/2}l^{2m}}{k^{2q-1}}\int_0^K \frac{sn^{2n}u\, cn^{2n}u\, dn^{2q}u\, du}{(1-l^2sn^2u)^m}. \tag{3.20}
$$

When k^2 and l^2 lie in their cut planes, the path of integration in the u-plane corresponding to $0 \leqq \phi \leqq \pi$ is a simple curve which does not pass through any singularities of the integrand. For values of k^2 and l^2 in the cut planes such that $k^2 \neq l^2$, $k \neq 0$, we see that (3.20) can be expressed in terms of complete elliptic integrals of the first, second, and third kinds. When $k^2 = l^2$, and hence, $x = 0$, only complete elliptic integrals of the first and second kinds occur. Finally, when either b or y vanishes, the integrand reduces to $\sin^{2n}u \cos^{2n}u$, and the upper limit becomes $\pi/2$.

Let

$$
\bar{K}(m,n) = \int_0^K \frac{sn^{2n}u\, cn^{2n}u\, du}{(1-l^2sn^2u)^m}; \tag{3.21}
$$

$$
L(m,n) = \int_0^K \frac{sn^{2n}u\, cn^{2n}u\, du}{dn^{2m}u}. \tag{3.22}
$$

Then when $q \geqq 0$, we have

$$
K(q,m,n) = \frac{2^{2n-m+q+1/2}k^{2m-2q+1}l^{2m-2q}}{(l^2-k^2)^{m-q}}\sum_{i=m-q}^{m}\binom{q}{m-i}\frac{(l^2-k^2)^i}{k^{2i}}\bar{K}(i,n), \tag{3.23}
$$

and when $q < 0$, we find on expanding $du^{2q}u(1 - l^2sn^2u)^{-m}$ in partial fractions that

$$
\begin{aligned}
K(q,m,n) &= \frac{2^{2n-m+q+1/2}k^{2m-2q+1}l^{2m-2q}}{(l^2-k^2)^{m-q}} \\
&\quad \cdot \left\{\sum_{i=1}^{-q}(-1)^{m+q+i}\binom{-m}{-q-i}\frac{(l^2-k^2)^i}{l^{2i}}L(i,n)\right. \\
&\qquad\qquad \left. + \sum_{i=1}^{m}\binom{q}{m-i}\frac{(l^2-k^2)^i}{k^{2i}}\bar{K}(i,n)\right\}.
\end{aligned} \tag{3.24}
$$

We see that (3.21) and (3.22) can be expressed in terms of integrals of the forms

$$(3.25)\qquad U_n = \int_0^K (1 - l^2 sn^2 u)^n \, du, \qquad V_n = \int_0^K (1 - l^2 sn^2 u)^{-n} \, du$$

and

$$(3.26)\qquad u_n = \int_0^K dn^{2n} u \, du, \qquad v_n = \int_0^K dn^{-2n} u \, du,$$

respectively, where $n \geqq 0$, and the latter can be evaluated by use of well-known recurrence relations [19, equations (331.03), (336.03), (314.05) and (315.05)].

Substituting $q = \frac{1}{2}$ in (3.20) and letting k tend to zero with l fixed, we obtain

$$(3.27)\qquad J(m, n) = 2^{2n-m+1} l^{2m} \int_0^{\pi/2} \frac{\sin^{2n} u \cos^{2n} u \, du}{(1 - l^2 \sin^2 u)^m}.$$

Thus, an alternative method for the evaluation of $J(m, n)$ is to express (3.27) in terms of the integrals

$$(3.28)\qquad g_n = \int_0^{\pi/2} (1 - l^2 \sin^2 u)^n \, du, \qquad h_n = \int_0^{\pi/2} (1 - l^2 \sin^2 u)^{-n} \, du,$$

and to evaluate the latter by means of recurrence relations obtained from those for U_n and V_n, respectively.

4. Recurrence Relations for $I_1(m, \nu)$ and $I_2(m, \nu)$. Using the identities

$$(4.1)\qquad \frac{d}{dz} J_\nu(z) = \frac{\nu}{z} J_\nu(z) - J_{\nu+1}(z)$$

and

$$(4.2)\qquad \frac{d}{dz} J_\nu(z) = J_{\nu-1}(z) - \frac{\nu}{z} J_\nu(z),$$

and assuming the conditions on x, y, and b mentioned earlier, we have

$$(4.3)\qquad \begin{aligned} y \int_0^\infty e^{-xt} J_{\nu-1}(bt) \left[J_{\nu-2}(yt) - \frac{(\nu - 1) J_{\nu-1}(yt)}{yt} \right] t^{m+1} \, dt \\ = \int_0^\infty e^{-xt} t^{m+1} J_{\nu-1}(bt) \frac{\partial}{\partial t} J_{\nu-1}(yt) \, dt, \end{aligned}$$

$$(4.4)\qquad \begin{aligned} b \int_0^\infty e^{-xt} J_{\nu-1}(yt) \left[\frac{(\nu - 1) J_{\nu-1}(bt)}{bt} - J_\nu(bt) \right] t^{m+1} \, dt \\ = \int_0^\infty e^{-xt} t^{m+1} J_{\nu-1}(yt) \frac{\partial}{\partial t} J_{\nu-1}(bt) \, dt, \end{aligned}$$

and

$$(4.5)\qquad \begin{aligned} b \int_0^\infty e^{-xt} J_\nu(yt) \left[J_{\nu-1}(bt) - \frac{\nu J_\nu(bt)}{bt} \right] t^{m+1} \, dt \\ = \int_0^\infty e^{-xt} t^{m+1} J_\nu(yt) \frac{\partial}{\partial t} J_\nu(bt) \, dt. \end{aligned}$$

We note that any other limits of integration can be used in the preceding as well as 0 and ∞. Integrating the right-hand sides of (4.3), (4.4) and (4.5) by parts, and using (4.1) in the first two cases and (4.2) in the last, we obtain the relations

$$\begin{aligned} bI_2(m, \nu) &= yI_2(m, \nu - 1) + (m + 1)I_1(m, \nu - 1) - xI_1(m + 1, \nu - 1) \\ &\quad + \begin{cases} 0, & \operatorname{Re}(2\nu + m - 1) > 0 \\ \dfrac{(by)^{\nu-1}}{2^{2\nu-2}[\Gamma(\nu)]^2}, & 2\nu + m - 1 = 0, \end{cases} \end{aligned} \tag{4.6}$$

$$\begin{aligned} &(2\nu + m - 1)I_1(m, \nu - 1) - bI_2(m, \nu) - xI_1(m + 1, \nu - 1) \\ &\quad = y \int_0^\infty e^{-xt} J_\nu(yt) J_{\nu-1}(bt) t^{m+1}\, dt - \begin{cases} 0, & \operatorname{Re}(2\nu + m - 1) > 0 \\ \dfrac{(by)^{\nu-1}}{2^{2\nu-2}[\Gamma(\nu)]^2}, & 2\nu + m - 1 = 0, \end{cases} \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} &(2\nu - m - 1)I_1(m, \nu) - yI_2(m, \nu) + xI_1(m + 1, \nu) \\ &\quad = b \int_0^\infty e^{-xt} J_\nu(yt) J_{\nu-1}(bt) t^{m+1}\, dt, \qquad \operatorname{Re}(2\nu + m + 1) > 0, \end{aligned} \tag{4.8}$$

respectively. Multiplying (4.7) by b and (4.8) by y and subtracting, we obtain

$$\begin{aligned} (y^2 - b^2)I_2(m, \nu) &= (2\nu - m - 1)yI_1(m, \nu) + xyI_1(m + 1, \nu) \\ &\quad -(2\nu + m - 1)bI_1(m, \nu - 1) + xbI_1(m + 1, \nu - 1) \\ &\quad - \begin{cases} 0, & \operatorname{Re}(2\nu + m - 1) > 0 \\ \dfrac{b(by)^{\nu-1}}{2^{2\nu-2}[\Gamma(\nu)]^2}, & 2\nu + m - 1 = 0. \end{cases} \end{aligned} \tag{4.9}$$

Finally, from (4.6) and (4.9), we find the recurrence relation

$$\begin{aligned} (2\nu - m + 1)byI_1(m, \nu + 1) &= 2\nu(y^2 + b^2)I_1(m, \nu) \\ &\quad - bxyI_1(m + 1, \nu + 1) - (2\nu + m - 1)byI_1(m, \nu - 1) \\ &\quad + bxyI_1(m + 1, \nu - 1) - \begin{cases} 0, & \operatorname{Re}(2\nu + m - 1) > 0 \\ \dfrac{(by)^{\nu+1}}{2^{2\nu}[\Gamma(\nu + 1)]^2}, & 2\nu + m - 1 = 0 \end{cases} \end{aligned} \tag{4.10}$$

for $I_1(m, \nu)$ alone.

It is easily verified that when $I_1(m, n)$ is known for $2n + m = 0$, 1, and 2, $m \leqq -1$, and when $I_1(0, n)$ is known for $n \geqq 0$, we can obtain $I_1(m, n)$ for all other values of $m < 0$ and $n \geqq 0$ for which it converges by means of (4.10). We can then obtain $I_2(m, n)$ from (4.9). Also, when $I_1(m, 0)$ and $I_1(m, 1)$ are known for $m \geqq 0$, and when $I_1(0, n)$ is known for $n \geqq 0$, we can calculate $I_1(m, n)$ for all other values of $m > 0$ and $n > 1$ by solving (4.10) for $I_1(m + 1, n + 1)$. As before, we can then obtain $I_2(m, n)$ from (4.9).

U. S. Naval Ordnance Laboratory
White Oak, Silver Spring, Maryland

1. A. WEINSTEIN, "On axially symmetric flows," *Quart. Appl. Math.*, v. 5, 1948, p. 429–444.
2. A. WEINSTEIN, "On the torsion of shafts of revolution," *Proc. VIIth Int. Congr. Appl. Mech.*, v. 1, 1948, p. 108.
3. H. BATEMAN, *Partial Differential Equations of Mathematical Physics*, Dover, New York, 1944.
4. A. G. WEBSTER, *Partial Differential Equations of Mathematical Physics*, Hafner, New York, 1947.
5. G. EASON, B. NOBLE & I. N. SNEDDON, "On certain integrals of Lipschitz-Hankel type involving products of Bessel functions," *Philos. Trans. Roy. Soc.*, v. 247, 1955, p. 529–551.
6. Y. L. LUKE, *Integrals of Bessel Functions*, McGraw-Hill, New York, 1962.
7. W. M. HICKS, "On toroidal functions," *Philos. Trans. Roy. Soc.*, v. 171, 1881, p. 609–652.
8. G. M. MINCHIN, "The magnetic field of a circular current," *Philos. Mag.*, v. 35, 1893, p. 354–365.
9. A. VAN TUYL, "On the axially symmetric flow around a new family of half-bodies," *Quart. Appl. Math.*, v. 7, 1950, p. 399–409.
10. M. R. SHURA-BURA, "Calculation of an integral containing a product of Bessel functions," *Dokl. Akad. Nauk. SSSR.*, v. 73, 1950, p. 901–903.
11. M. A. SADOWSKY & E. STERNBERG, "Elliptic integral representations of axially symmetric flows," *Quart. Appl. Math.*, v. 8, 1950, p. 113–126.
12. G. E. PRINGLE, "The potential of a circular current," *Proc. Cambridge Philos. Soc.*, v. 57, 1961, p. 385–392.
13. H. E. FETTIS, "Note on $\int_0^\infty e^{-x}J_0(\eta x/\xi)J_1(x/\xi)x^{-n}\,dx$," *Math. Comp.*, v. 14, 1960, p. 372–374.
14. G. P. WEEG, "Numerical Integration of $\int_0^\infty e^{-x}J_0(\eta x/\xi)J_1(x/\xi)x^{-n}\,dx$," *MTAC*, v. 13, 1959, p. 312–313.
15. A. WEINSTEIN, "Discontinuous integrals and generalized potential theory," *Trans. Amer. Math. Soc.*, v. 63, 1948, p. 342–354.
16. G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, Second Edition, Cambridge University Press, 1944.
17. A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER & F. G. TRICOMI, *Higher Transcendental Functions*, Vol. 1, McGraw-Hill, New York, 1953.
18. E. W. HOBSON, *The Theory of Spherical and Ellipsoidal Harmonics*, Cambridge University Press, 1931.
19. P. F. BYRD & M. D. FRIEDMAN, *Handbook of Elliptic Integrals for Engineers and Physicists*, Springer, Berlin, 1954.