Some Direct Decompositions of the Set of Integers

By N. G. de Bruijn

1. Introduction. Every positive integer can be uniquely represented in the form

$$x = \epsilon_0 + 2\epsilon_1 + 2^2\epsilon_2 + 2^3\epsilon_3 + \cdots$$

with $\epsilon_i = 0$ or 1, $\epsilon_i = 0$ from a certain point onwards, and the sequence $1, 2, 2^2, \cdots$ is essentially the only one with this property. The situation is entirely different, however, if we require that all integers, positive, negative or zero, be represented. Some of the simplest possibilities are obtained by replacing $1, 2, 2^2, 2^3, \cdots$ by $1, -2, 2^2, -2^3, 2^4, -2^5, \cdots$ or by $1, 2, -2^2, 2^3, 2^4, -2^5, 2^6, 2^7, -2^8, \cdots$, but there are very many others. Such decompositions were studied in [1]. In particular, cases were investigated where the sequence has the form $M, -2N, 2^2M, -2^3N, 2^4M, -2^5N, \cdots$. It is to this type of decomposition of the set of integers that the present paper is devoted entirely.

It is natural to split the resulting representation of x according to terms with M and terms with N, so $x = Ms_1 - 2Ns_2$, and now we can reformulate the problem in terms of the set S of all possible values that s_1 and s_2 can assume.

The set S is defined as the set of all nonnegative integers with the property that, when represented in the scale of 4, they do not contain 2's or 3's, but only 0's and 1's. So

$$S = \{0, 1, 4, 5, 16, 17, 20, 21, 64, 65, 68, 69, 80, 81, 84, 85, 256, 257, 260, 261, 272, 273, 276, 277, 320, 321, \cdots\}.$$

Let X denote the set of all integers, positive or negative or zero. The decompositions of X we shall be concerned with are of the type illustrated by the following examples: every $x \in X$ can be represented in exactly one way as

$$x = s_1 - 2s_2 \qquad (s_1 \in S, s_2 \in S),$$

and in exactly one way as

$$x = 7s_1 - 2s_2 \qquad (s_1 \in S, s_2 \in S).$$

We shall say that a pair (M, N) of nonnegative integers is a *good pair* if it is true that every $x \in X$ can be represented uniquely as

$$x = Ms_1 - 2Ns_2 \qquad (s_1 \in S, s_2 \in S).$$

Saying that (M, N) is good is equivalent to saying that the formal relation

$$F(z^M)\, F(z^{-2N}) = \sum_{k=-\infty}^{\infty} z^k$$

holds, where

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$$F(z) = (1 + z)(1 + z^4)(1 + z^{16})(1 + z^{64}) \cdots .$$

These good pairs were discussed in [1, Section 4] (there they were called "basic" pairs), as a special case in a more general setting. The following statements were proved there:

(1) If (M, N) is good, then (N, M) is good.

(2) If either M or N is divisible by any number of the form $2^m + 1$ ($m = 0, 1, 2, \cdots$), then (M, N) is not good.

(3) For each k ($k = 0, 1, 2, \cdots$), the pair $(1, 2^{2k+1} - 1)$ is good.

(4) If (M, N) is good, then $M \equiv N \pmod 6$.

(5) If (M, N) is good, then M and N have no common divisor.

(6) If (M, N) is good, and $t = N/M$, then t *does not belong* to any of the following intervals:

$$1 < t < \tfrac{3}{2}, \quad 3 < t < 6, \quad 11 < t < 24, \quad 43 < t < 96, \quad \cdots .$$

If we use the fact that either $M = 1$ or $M \geqq 7$ (see (2)), the argument can be refined a little, and leads to the conclusion that t does not belong to any of the intervals $2^{2k+1}/3 \leqq t \leqq 3 \cdot 2^{2k-1}$ ($k = 0, \pm 1, \pm 2, \cdots$).

There exists a procedure by which, for each pair (M, N), we can decide in a finite number of steps whether it is good or not. It is a special case of a procedure for a slightly more general situation (see [1]), and can be described as follows. We construct an oriented graph whose vertices are the integers. If $x \in X$, $x_1 \in X$, we take an oriented edge from x to x_1 if and only if one of the following relations holds:

$$x = 4x_1, \quad x = 4x_1 + M, \quad x = 4x_1 - 2N, \quad x = 4x_1 + M - 2N.$$

Assuming that both M and N are odd (which is obviously necessary for the pair to be good), we notice that to each x there belongs exactly one x_1.

Removing the loop from 0 to 0, we have

(7) The pair (M, N) is good if and only if the graph is a tree (whose root is 0, of course).

We need to investigate only the part of the graph lying in the interval

$$\text{(A)} \qquad -M/3 \leqq x \leqq 2N/3,$$

for if $x > 2N/3$ we have $-M/3 < x_1 < x$, and if $x < -M/3$ we have $x < x_1 < 2N/3$. So if the part inside (A) is a tree, the whole graph is a tree. We denote the part inside (A) by Γ.

In [1] we listed all good pairs as far as $1 \leqq M \leqq N \leqq 100$, obtained with the aid of pencil and paper. (This included making a table of the relation between x and x_1, constructed with four strips of paper that simply had to be shifted in order to switch on the next pair.) The material has now been extended considerably with the assistance of an IBM 1620 computer. The author is indebted to Mrs. E. Simarro who did a large part of the actual programming.

2. Computations and Observations. A pair (M, N) was investigated as follows. We consider the component Γ_1 of Γ that contains 0. This Γ_1 is always a tree, and the question is whether $\Gamma_1 = \Gamma$. In other words, the question is whether the number of vertices of Γ_1 is equal to the total number of integers in the interval $-M/3 \leqq$

$x \leqq 2N/3$. The number of vertices of Γ_1 was obtained by tracing Γ_1 in such a way that each edge is taken just once in one direction, and just once in the other direction. At each point investigated during the tracing, we kept a record of the direct path leading from that point to 0.

In Table 1 we list all good pairs as far as M and N are at most 1800. According to (1) we restricted ourselves to $M \leqq N$, and the pairs are arranged with increasing N. If either N or M was divisible by 2, 3 or 5 (see (2)), or if $N - M$ was not a multiple of 6 (see (4)), the pair was not investigated any further. Also, the pair was dropped if $N/24 \leqq M \leqq 3N/32$, or $N/6 \leqq M \leqq 3N/8$ or $2N/3 \leqq M \leqq N$ (see (6)).

Table 1 shows many pairs with $M = 1$ (by (3) we know there are infinitely many of them). In a separate investigation we determined all $N \leqq 24889$ such that $(1, N)$ is good. These are shown in Table 2. The gaps predicted by (6) are clearly visible: between 7 and 31, between 37 and 109, between 679 and 1579, between 2677 and 6493, and between 10879 and 24649. For *all* $N \leqq 24889$ (restricted to $N \equiv 1 \pmod 6$, $N \not\equiv 0 \pmod 5$) we determined the number of vertices of the subtree Γ_1, but these numbers are not listed in this paper. Inside the gaps as indicated by (7), this number of vertices is constant, and in fact, the tree undergoes deformations only if N runs through such intervals, and the topological structure is unaltered. Γ_1 has 2 vertices if $N = 13$ or 19, 4 vertices if $43 \leqq N \leqq 91$, 8 vertices if $181 \leqq N \leqq 379$, 16 vertices if $691 \leqq N \leqq 1531$, 32 vertices if $2731 \leqq N \leqq 6139$, 64 vertices if $10927 \leqq N \leqq 24571$. This is easily proved; we do not need a computer for this.

There are also other gaps in the list, again corresponding to trees with constant shape, but slightly more complicated than in the cases just mentioned. For example, there are 40 vertices if $1777 \leqq N \leqq 1879$, 48 vertices if $2221 \leqq N \leqq 2359$, 80 vertices if $7111 \leqq N \leqq 7559$, 96 vertices if $8881 \leqq N \leqq 9451$.

Another look at Table 1 shows that there are many good pairs where $N = 2M + 1$ or $N = 2M - 1$. These were investigated separately up to $M = 8171$, and listed in Table 3. Also here, we notice a number of gaps, which can easily be discussed theoretically: if $4^k < M < 6\cdot 4^{k-1}$, $N = 2M \pm 1$, then the subtree Γ_1 has exactly 2^k vertices.

Some good pairs look particularly pretty. For example:

$$(7, 13), (7, 13^2); \quad (7^2, 11^2), (1, 11^2), (1, 23^2); \quad (1, 7), (1, 7^2), (1, 7^4).$$

A more careful inspection of the available material led to the discovery of some infinite sequences of good pairs. The pairs

$$(11, 23), (11, 89), (11, 353), (11, 1409),$$

all of the form $(11, 22\cdot 4^k + 1)$, can be found in Table 1, and the computer proved that the next three items, viz. (11, 5633), (11, 22529), (11, 90113) are also good. A general proof is presented below (Theorem 1).

Another remarkable sequence can be conjectured from Table 2, viz. $(1, 10\cdot 4^k - 3)$, of which Table 2 shows the first six cases, viz.

$$(1, 7), (1, 37), (1, 157), (1, 637), (1, 2557), (1, 10237).$$

See Theorem 2 for a general proof.

TABLE 1

Good pairs with $1 \leqq M \leqq N \leqq 1800$

M	N	M	N	M	N	M	N	M	N
1	1	7	277	311	539	373	781	521	1043
1	7	139	277	293	563	511	781	683	1043
7	13	47	287	373	571	397	787	131	1049
11	23	143	287	91	583	397	793	667	1051
1	31	151	301	343	583	311	803	31	1057
13	31	31	307	23	587	491	809	131	1061
19	31	161	317	59	587	511	817	7	1063
1	37	211	319	367	589	499	823	133	1063
7	43	163	331	293	593	349	829	131	1067
31	49	13	337	23	599	7	841	103	1069
31	61	127	337	347	599	7	853	11	1073
31	67	43	343	371	599	83	857	113	1073
47	71	199	343	19	601	367	859	29	1079
7	73	211	343	61	607	547	877	539	1079
31	73	11	347	317	611	413	881	157	1081
41	77	173	347	287	617	7	883	679	1087
47	77	11	353	19	619	383	887	167	1091
41	83	179	359	401	623	437	887	109	1093
11	89	157	367	1	631	467	887	529	1093
49	97	229	367	1	637	139	889	547	1093
41	101	181	379	61	637	541	889	137	1097
49	103	61	397	419	641	419	893	667	1099
1	109	157	397	61	643	7	907	443	1103
1	121	53	401	299	647	109	919	107	1109
49	121	203	401	1	661	389	923	137	1109
79	121	1	403	97	661	31	931	139	1111
1	127	203	407	311	671	541	931	43	1117
79	127	193	409	413	671	7	937	7	1123
1	133	217	409	419	671	91	937	589	1123
89	143	11	419	349	673	7	943	691	1123
1	151	1	421	421	673	97	943	167	1127
79	151	199	427	293	677	553	943	701	1127
1	157	211	427	1	679	491	947	457	1129
103	157	217	433	107	683	31	949	719	1133
89	167	181	451	23	701	511	961	37	1141
7	169	229	451	371	701	367	967	133	1147
7	193	67	457	83	713	457	967	151	1153
91	193	287	461	89	719	491	971	463	1153
121	193	47	467	107	719	517	973	571	1159
31	199	19	469	73	727	31	979	37	1171
121	199	73	469	97	733	373	979	463	1171
89	209	251	473	307	733	491	983	469	1171
101	209	59	479	451	733	157	997	589	1177
131	209	193	487	121	739	167	1007	733	1177
7	211	263	497	23	749	499	1009	149	1187
103	217	199	499	403	757	161	1019	587	1187
109	217	1	511	19	763	401	1019	161	1199
7	223	61	511	127	763	37	1021	599	1199
31	229	67	511	463	763	511	1021	751	1201
137	233	247	511	77	767	511	1027	611	1211
127	253	209	521	451	769	137	1031	713	1211
163	253	1	529	503	773	43	1033	499	1213
29	269	83	539	337	781	667	1033	647	1217

TABLE 1—*Continued*

M	N	M	N	M	N	M	N	M	N
49	1219	211	1369	149	1493	997	1603	733	1693
571	1219	43	1381	719	1499	803	1607	647	1697
589	1219	683	1391	749	1499	199	1609	1109	1697
149	1247	139	1393	59	1511	767	1613	727	1699
479	1247	11	1409	203	1523	1	1621	53	1703
479	1253	707	1409	983	1529	193	1621	173	1703
13	1267	217	1417	721	1537	763	1621	1061	1703
149	1271	47	1421	907	1537	869	1631	11	1709
161	1283	749	1427	893	1541	1	1633	851	1709
31	1291	547	1429	181	1543	203	1637	1061	1721
41	1307	47	1433	917	1559	857	1637	11	1733
623	1307	167	1433	181	1561	667	1639	271	1741
167	1319	851	1433	817	1561	809	1643	283	1753
647	1319	581	1451	193	1567	661	1651	11	1757
821	1319	737	1451	197	1571	791	1661	167	1757
823	1321	583	1453	971	1571	13	1663	911	1757
133	1327	197	1457	743	1577	13	1669	73	1759
691	1333	173	1463	791	1577	259	1669	211	1771
137	1349	587	1463	1	1579	251	1679	937	1771
199	1351	589	1471	803	1583	677	1679	1123	1771
517	1351	199	1477	631	1591	223	1687	1091	1787
43	1357	11	1481	991	1597	53	1691	13	1789
683	1367	737	1481	817	1603	899	1691	899	1793
163	1369								

TABLE 2

Values of N (with $1 \leqq N \leqq 24889$*) such that (1, N) is a good pair*

1	151	661	2041	2173	6661	7591	8089	8311	9487	9991	10669
7	157	679	2047	2401	6703	7717	8101	8317	9493	10087	10741
31	403	1579	2053	2527	6733	7729	8173	8497	9601	10111	10837
37	421	1621	2071	2557	6871	7747	8191	8569	9631	10159	10879
109	511	1633	2077	2677	6967	7753	8197	8623	9757	10237	24649
121	529	1969	2143	6493	6973	7819	8221	8701	9937	10261	24751
127	631	1981	2149	6559	6979	7861	8257	9457	9961	10663	24781
133	637	2017	2167	6643	7009						

TABLE 3

Good pairs (M, N) with $N = 2M \pm 1$; $1 \leqq M \leqq 8171$

M	N	M	N	M	N	M	N	M	N
1	1	143	287	521	1043	989	1979	2723	5447
7	13	151	301	539	1079	2041	4081	2761	5521
11	23	173	347	547	1093	2047	4093	2999	5999
31	61	179	359	589	1177	2107	4213	3661	7321
41	83	203	407	599	1199	2153	4307	3721	7441
49	97	217	433	683	1367	2191	4381	3739	7477
109	217	397	793	749	1499	2219	4439	6199	12397
127	253	491	983	803	1607	2531	5063	6923	13847
139	277	511	1021	929	1859	2693	5387	7949	15899

In Table 3 we can discover the sequence

$$(1, 1), (7, 13), (31, 61), (127, 253), (511, 1021), (2047, 4093),$$

i.e., the first six items of the sequence $(2^{2k-1} - 1, 2^{2k} - 3)$. See Theorem 3 for a general proof.

The case (11, 90113) was the "largest" one presented to the computer during this investigation. In that case we have a very large tree, filling the whole interval $[-30037, +7]$ (see (A)), and it took the computer 14 minutes to make sure that the tree has 30045 nodes indeed. The "height" of this tree is comparatively small: its highest point -21838 has level 32, i.e., iteration of the mapping $x \to x_1$, starting with $x = -21838$, produces 0 in 32 steps: $-21838 \to -5454 \to -1358 \to -334 \to -78 \to -14 \to +2 \to +6 \to +7 \to -22521 \to -28153 \to -29561 \to -29913 \to -30001 \to -30023 \to -30034 \to -7503 \to -24404 \to -6101 \to -24048 \to -6012 \to -1503 \to -22904 \to -5726 \to -1426 \to -351 \to -22616 \to -5654 \to -1408 \to -352 \to -88 \to -22 \to 0$.

3. Three Theorems. As a preparation for Theorems 1, 2, 3, we first prove two lemmas.

LEMMA 1. *If a pair (M, N) is not a good pair then there exists a positive integer n, and there exist numbers $\epsilon_1, \cdots, \epsilon_n, \delta_1, \cdots, \delta_n$, all either 0 or 1, but not all 0, such that*

$$\sum_{i=1}^{n} (M\epsilon_i - 2N\delta_i)4^{i-1} \equiv 0 \pmod{4^n - 1}.$$

Proof (cf. [1], Theorem 7). It follows from (7) that if Γ is not a tree, then the interval (A) has to contain a closed cycle of Γ, viz. $x = x_0 \to x_1 \to x_2 \to \cdots \to x_n = x_0$ and $x_i = 4x_{i+1} + M\epsilon_i - 2N\delta_i$, with suitable values of ϵ_i and δ_i. It follows that

$$\sum_{i=1}^{n} (M\epsilon_i - 2N\delta_i)4^{i-1} = -(4^n - 1)x_0.$$

Since the loop $0 \to 0$ was removed from our graph, we have $x_0 \neq 0$; this excludes $\epsilon_1 = \cdots = \epsilon_n = \delta_1 = \cdots = \delta_n = 0$.

LEMMA 2. *Let $n, k, \epsilon_1, \cdots, \epsilon_n$ be integers ($n \geqq 1$). If p is an integer, we define $\{p\}$ as the number defined by $\{p\} \equiv p \pmod{n}$, $0 < \{p\} \leqq n$. Then we have*

$$4^k \sum_{i=1}^{n} \epsilon_i\, 4^{i-1} \equiv \sum_{i=1}^{n} \epsilon_{\{i-k\}}\, 4^{i-1} \pmod{4^n - 1}.$$

Proof. We have $4^{k+i-1} \equiv 4^{\{k+i\}-1}$; moreover

$$\sum_{i=1}^{n} \epsilon_{\{i\}}\, 4^{\{k+i\}-1} = \sum_{i=k+1}^{n+k} \epsilon_{\{i-k\}}\, 4^{\{i\}-1} = \sum_{i=1}^{n} \epsilon_{\{i-k\}}\, 4^{\{i\}-1}.$$

LEMMA 3. *Let $n, \zeta_1, \cdots, \zeta_n$ be integers, and assume that*

$$\sum_{i=1}^{n} \zeta_i\, 4^{i-1} \equiv 0 \pmod{4^n - 1}.$$

Then there exist integers $t_0, t_1, \cdots, t_n = t_0$ such that

$$\tfrac{1}{3}\min_i \zeta_i \leqq t_j \leqq \tfrac{1}{3}\max_i \zeta_i \qquad (j = 0, \cdots, n)$$
$$t_{j+1} = 4t_j - \zeta_{\{-j\}} \qquad (j = 0, \cdots, n-1).$$

Proof. For each j we put

$$w_j = \sum_{i=1}^{n} \zeta_{\{i-j\}}\, 4^{i-1},$$

and we easily obtain $w_{j+1} = 4w_j - (4^n - 1)\,\zeta_{\{-j\}}$. Furthermore we have

$$w_j \leqq (\max_i \zeta_i)(1 + 4 + \cdots + 4^{n-1}) = \tfrac{1}{3}(\max_i \zeta_i)(4^n - 1),$$

and a similar lower estimate. So taking $t_j = w_j/(4^j - 1)$, we have proved our lemma.

THEOREM 1. *If* $k = 0, 1, 2, \cdots$, *the pair* $(22\cdot 4^k + 1, 11)$ *is good.*

Proof. Assume that this is false for some k. Apply Lemma 1, with $M = 22\cdot 4^k + 1$, $N = 11$; this produces $\epsilon_1, \cdots, \epsilon_n, \delta_1, \cdots, \delta_n$, all 0 or 1, and not all 0. We split M into the two parts 1 and $22\cdot 4^k$; to the second part we apply Lemma 2. Thus we obtain

$$\sum_{i=1}^{n} (\epsilon_i + 22(\epsilon_{\{i-k\}} - \delta_i))4^{i-1} \equiv 0 \pmod{4^n - 1}.$$

To this sum we apply Lemma 3, with $\zeta_i = \epsilon_i + 22\,(\epsilon_{\{i-k\}} - \delta_i)$. That produces the cycle $t_0, t_1, \cdots, t_n = t_0$. We have $\zeta_i \in Z = \{0, 22, -22, 1, 23, -21\}$, so by Lemma 3 the t's are restricted by $-7 \leqq t_j \leqq 7$, $t_{j+1} - 4t_j \in Z$. We can now make a list of all possibilities for the vector $(t_j, t_{j+1}, \zeta_{\{-j\}})$. They are

$(-7, -7, -21)$, $(-7, -6, -22)$, $(-6, -3, -21)$, $(-6, -2, -22)$,
$(-5, 1, -21)$, $(-5, 2, -22)$, $(-4, 5, -21)$, $(-4, 6, -22)$, $(-1, -5, 1)$,
$(-1, -4, 0)$, $(0, -1, 1)$, $(0, 0, 0)$, $(1, 3, 1)$, $(1, 4, 0)$, $(2, 7, 1)$, $(4, -7, 23)$,
$(4, -6, 22)$, $(5, -3, 23)$, $(5, -2, 22)$, $(6, 1, 23)$, $(6, 2, 22)$, $(7, 5, 23)$,
$(7, 6, 22)$.

The possible transitions from a t_j to the next one, t_{j+1}, can be visualized by drawing a graph with the nodes $-7, \cdots, +7$, whose oriented edges represent the transitions $t_j \to t_{j+1}$ that can be obtained from the above vectors: $(-7, -7)$, $(-7, -6)$, $(-6, -3)$, $(-6, -2)$, $(-5, 1)$, $(-5, 2)$, $(-4, -5)$, $(-4, -6)$, $(-1, -5)$, $(-1, -4)$, $(0, -1)$, $(0, 0)$, $(1, 3)$, $(1, 4)$, $(2, 7)$, $(4, -7)$, $(4, -6)$, $(5, -3)$, $(5, -2)$, $(6, 1)$, $(6, 2)$, $(7, 5)$, $(7, 6)$. The cycle $t_0 \to t_1 \to \cdots \to t_n = t_0$ represents some closed circuit in this oriented graph. It is easy to see that there can be only the following three closed circuits: (a) $\cdots \to 0 \to 0 \to 0 \to \cdots$, (b) $\cdots \to -7 \to -7 \to -7 \to \cdots$, (c) $\cdots \to 2 \to 7 \to 6 \to 2 \to 7 \to 6 \to \cdots$; in case (c) it is obvious that n is a multiple of 3.

In case (a) we have $t_j = t_{j+1} = 0$ for all j, whence $\zeta_i = 0$ for all i. By $\zeta_i = \epsilon_i + 22(\epsilon_{\{i-k\}} - \delta_i)$ we deduce that $\epsilon_1 = \cdots = \epsilon_n = \delta_1 = \cdots = \delta_n = 0$. This contradicts our assumption, so case (a) is impossible.

In case (b) we have $t_j = t_{j+1} = -7$ for all j, whence $\zeta_i = -21$ for all j, so

$\epsilon_i = 1$, $\epsilon_{\{i-k\}} = 0$, $\delta_i = 1$ for all i. This is impossible, for if $\epsilon_i = 1$ for all i then $\epsilon_{\{i-k\}} = 1$ for all i. This excludes case (b).

We finally consider case (c). The transitions $2 \to 7$, $7 \to 6$, $6 \to 2$ correspond to the vectors $(2, 7, 1)$, $(7, 6, 22)$, $(6, 2, 22)$, respectively. If $\zeta_i = 1$ we have $\epsilon_i = 1$, $\epsilon_{\{i-k\}} - \delta_i = 0$, if $\zeta_i = 22$ we have $\epsilon_i = 0$, $\epsilon_{\{i-k\}} - \delta_i = 1$. Therefore, the average value of ϵ_i is $\frac{1}{3}$, and the average of $\epsilon_{\{i-k\}} - \delta_i$ is $\frac{2}{3}$. It follows that the average of δ_i is $-\frac{1}{3}$, which is impossible. This excludes (c), which was the one last possibility, and our proof is complete.

THEOREM 2. *If* $k = 0, 1, 2, \cdots$, *the pair* $(10 \cdot 4^k - 3, 1)$ *is good.*

Proof. The proof is similar to the one of Theorem 1. Assume that, for some k, $(10 \cdot 4^k - 3, 1)$ is not good. Apply Lemma 1, with $M = 10 \cdot 4^k - 3$, $N = 1$; this again produces $\epsilon_1, \cdots, \epsilon_n, \delta_1, \cdots, \delta_n$, all 0 or 1, and not all 0. We obtain, using Lemma 2,

$$\sum_{i=1}^{n} (-3\epsilon_i + 10\epsilon_{\{i-k\}} - 2\delta_i)4^{i-1} \equiv 0 \pmod{4^n - 1}.$$

We apply Lemma 3 with $\zeta_i = -3\epsilon_i + 10\epsilon_{\{i-k\}} - 2\delta_i$. As in the proof of Theorem 1, we obtain $Z = \{0, 10, 8, -2, -3, 7, 5, -5\}$; the restriction on the elements of the cycle $t_0 \to t_1 \to \cdots \to t_n = t_0$ is $-1 \leqq t_j \leqq 3$, and the possible vectors $(t_j, t_{j+1}, \zeta_{\{-j\}})$ are

$$(-1, -1, -3), (-1, 1, -5), (0, 0, 0), (0, 2, -2), (0, 3, -3), (1, -1, 5),$$
$$(2, 0, 8), (2, 1, 7), (2, 3, 5), (3, 2, 10).$$

Drawing the transition graph with nodes $-1, 0, 1, 2, 3$ and oriented edges $(-1, -1)$, $(-1, 1)$, $(0, 0)$, $(0, 2)$, $(0, 3)$, $(1, -1)$, $(2, 0)$, $(2, 1)$, $(2, 3)$, $(3, 2)$, we notice that the t-cycle has to lie either in the subgraph determined by the nodes 0, 2, 3 or in the subgraph determined by $1, -1$, because there is no way leading from the second group to the first one.

First we assume that our cycle lies in the first group. For the transitions inside this group we evaluate $\tau_i = \epsilon_{\{i-k\}} - \epsilon_i$. Each value of $\zeta_i \in Z$ uniquely determines ϵ_i, $\epsilon_{\{i-k\}}$ and δ_i, and we obtain that $\tau_i = 0, 1, 1, 0, -1, 0, 0, -1$ if $\zeta_i = 0, 10, 8, -2, -3, 7, 5, -5$, respectively. It follows that the transitions $0 \to 0$, $0 \to 2$, $2 \to 0$, $0 \to 3$, $2 \to 3$, $3 \to 2$ correspond to $\tau_i = 0, 0, 1, -1, 0, 1$, respectively. The average of all τ_i has to be 0 (since ϵ_i and $\epsilon_{\{i-k\}}$ have the same average). It is obvious from the graph that if our cycle contains any transition other than $0 \to 0$, then it has to contain $2 \to 0$, where $\tau_i = 1$. This spoils the average: the value -1 on $0 \to 3$ cannot compensate for this, for $0 \to 3$ is necessarily followed by $3 \to 2$, and the joint contribution of $0 \to 3$ and $3 \to 2$ to the sum of the τ's is 0. Thus there only remains the cycle $\cdots \to 0 \to 0 \to 0 \to \cdots$. But this cycle leads to $\epsilon_1 = \cdots = \epsilon_n = \delta_1 = \cdots = \delta_n = 0$, which was excluded.

We next assume that our cycle lies in the second group, where we have the transitions $-1 \to -1$, $-1 \to 1$, $1 \to -1$. These correspond to the values $\tau_i = -1, -1, 0$, respectively. So it follows that in any cycle all τ_i are $\leqq 0$, some are < 0, and the average cannot be 0. This completes the proof.

THEOREM 3. *If* $k = 1, 2, 3, \cdots$, *the pair* $(2^{2k} - 3, 2^{2k-1} - 1)$ *is good.*

Proof. We follow the same pattern as in the proofs of the previous theorems.

The graph is much simpler in this case, but the reasoning has to be more delicate since the averaging argument fails.

Assuming that for some k the pair is not good, we obtain, using Lemmas 1 and 2

$$\sum_{i=1}^{n} (-3\delta_i + \delta_{\{i-k\}} + 2\epsilon_i - \epsilon_{\{i-k\}})4^{i-1} \equiv 0 \pmod{4^n - 1}.$$

We put

$$\epsilon_i - \delta_i = \sigma_i, \quad \zeta_i = 3\sigma_i - \sigma_{\{i-k\}} - \epsilon_i,$$

and we apply Lemma 3, producing a cycle $t_0, t_1, \cdots, t_n = t_0$. We have $4t_j - t_{j+1} = \zeta_{\{-j\}} \in Z$, and $Z = \{-4, -3, -2, -1, 0, 1, 2, 3\}$, whence $-1 \leqq t_j \leqq 1$. The possible vectors $(t_j, t_{j+1}, \zeta_{\{-j\}})$ are

$$(1, 1, 3), (0, 1, -1), (0, 0, 0), (0, -1, 1), (-1, 0, -4), (-1, -1, -3).$$

The only vector whose first entry equals 1 is $(1, 1, 3)$. So if $t_j = 1$ for one j, then $t_j = 1$ for all j, and $\zeta_{\{-j\}} = 3$ for all j. It follows that $\sigma_i = 1$ for all i, so $\sigma_{\{i-k\}} = 1$ for all i, whence $\epsilon_i = 3 - 1 - 3 = -1$ for all i. This is impossible. Henceforth we shall assume that $t_j \neq 1$ for all j.

First we take the case that all $t_j = 0$ for all j, so $\zeta_i = 0$ for all i. Hence $3\sigma_i = \sigma_{\{i-k\}} + \epsilon_i$ for all i; since $\sigma_j \in \{-1, 0, 1\}$, $\epsilon_j \in \{0, 1\}$ for all j, we deduce that $\sigma_i = \pm 1$ is impossible. So $\sigma_i = 0$ for all i, therefore $\sigma_{\{i-k\}} = 0$ for all i, hence $\epsilon_i = 0$ for all i. Therefore $\delta_i = \epsilon_i + \sigma_i = 0$ for all i, and we have reached $\epsilon_1 = \cdots = \epsilon_n = \delta_1 = \cdots = \delta_n = 0$, which was excluded beforehand.

Next we take the case that $t_j = -1$ for all j, so $\zeta_i = -3$ for all i. From the definition of ζ_i it follows that $\sigma_i = -1$ for all i. Therefore, $\sigma_{\{i-k\}} = -1$ for all i, and we arrive at $-3 = -3 + 1 - \epsilon_i$, so $\epsilon_i = 1$ for all i. Finally, $\delta_i = \epsilon_i - \sigma_i = 2$, but this is impossible since δ_i can be only 0 or 1.

It remains to investigate the case that some t_j are 0 and some are -1. Then there is a number m with $t_m = -1$, $t_{m+1} = 0$. Hence $\zeta_{\{-m\}} = -4$, and it follows that $\sigma_{\{-m\}} = -1$, hence $\epsilon_{\{-m\}} = 0$, $\delta_{\{-m\}} = 1$, hence $\sigma_{\{-m-k\}} = -\zeta_{\{-m\}} + 3\sigma_{\{-m\}} - \epsilon_{\{-m\}} = 1$. Thus we proved the existence of an index j with $\sigma_j = 1$.

From the fact that the t_i's only take the values 0 and -1 we deduce:

(B) If $\sigma_{\{-j\}} = 1$ then $t_j = 0$, $t_{j+1} = -1$, $\sigma_{\{-j-k\}} = 1$.

For, $\sigma_{\{-j\}} = 1$ excludes the values $\zeta_{\{-j\}} = -1, 0, -4, -3$, and this leaves for $(t_j, t_{j+1}, \zeta_{\{-j\}})$ only the possibility $(0, -1, 1)$. So $\zeta_{\{-j\}} = 1$; combined with $\sigma_{\{-j\}} = 1$ this leads to $\sigma_{\{-j-k\}} + \epsilon_{\{-j\}} = 2$, whence $\sigma_{\{-j-k\}} = 1$, $\epsilon_{\{-j\}} = 1$.

We also need:

(C) If $t_l = -1$, $t_{l+1} = 0$ then $\sigma_{\{-l-k\}} = 1$.

For, $\zeta_{\{-l\}} = 4t_l - t_{l+1} = -4$, whence $\sigma_{\{-l\}} = -1$, $\sigma_{\{-l-k\}} + \epsilon_{\{-l\}} = 1$. As $\epsilon_i = \sigma_i + \delta_i$, we have $\epsilon_{\{-l\}} \leqq \sigma_{\{-l\}} + 1 = 0$, whence $\epsilon_{\{-l\}} = 0$, $\sigma_{\{-l-k\}} = 1$.

We now finish the proof of the theorem. We know that at least once $\sigma_{\{-m\}} = -1$, and that at least once $t_j = 0$.

Let h be the smallest positive number such that m exists with

$$\sigma_{\{-m\}} = 1, \quad t_{m+h} = 0.$$

By (B) we have $t_{m+1} = -1$, whence $h > 1$. Therefore $t_{m+h-1} = -1$, because of the minimality of h.

From (B) we infer $\sigma_{\{-m-k\}} = 1$, and if we again apply (B), now with $j = m + k$, we obtain $t_{m+k} = 0$, $t_{m+k+1} = -1$.

Applying (C) with $l = m + h - 1$ we deduce $\sigma_{\{-m-h-k+1\}} = 1$. Next, applying (B) with $j = m + h + k - 1$, we find $t_{m+h+k-1} = 0$, $t_{m+h+k} = 1$. Thus we have obtained

$$\sigma_{\{-m-k\}} = 1, \quad t_{m+k+h-1} = 0, \quad h - 1 > 0,$$

and this contradicts the minimality property of h. The proof is now complete.

Technological University
Eindhoven, Netherlands

1. N. G. de Bruijn, "On bases for the set of integers," *Publ. Math. Debrecen*, v. 1, 1950 p. 232–242.