Triangle Formulas in the Complex Plane

By Philip J. Davis

1. Introduction. In the course of studying "simple" quadratures, the following identity was derived. Let T designate a triangle lying in the complex z plane whose vertices are z_1, z_2, z_3. If A designates the area of T and if f is any function that is regular in the closure of T, then

$$(1) \quad \frac{1}{2A} \iint_T f''(z)\, dx\, dy = \frac{f(z_1)}{(z_1 - z_2)(z_1 - z_3)} + \frac{f(z_2)}{(z_2 - z_1)(z_2 - z_3)} + \frac{f(z_3)}{(z_3 - z_1)(z_3 - z_2)}.$$

This formula can be put in the alternative form

$$(1') \quad \frac{1}{2A} \iint_T f''(z)\, dx\, dy = f(z_1, z_2, z_3) = \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ f(z_1) & f(z_2) & f(z_3) \end{vmatrix} : \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1^2 & z_2^2 & z_3^2 \end{vmatrix},$$

where $f(z_1, z_2, z_3)$ designates the second divided difference of f with respect to z_1, z_2, z_3. It may also be written as

$$(1'') \quad R(f; z_1) = \frac{(z_1 - z_2)(z_1 - z_3)}{2A} \iint_T f''(z)\, dx\, dy,$$

where $R(f; z_1)$ is the remainder at z_1 due to linear interpolation to $f(z)$ at z_2 and z_3.

This formula was subsequently found in *Approximation: Theory and Practice* by I. J. Schoenberg (Notes on a series of lectures at Stanford University, 1955) and correspondence with Schoenberg revealed that it was obtained by him and T. Motzkin in 1951–1952 during one of their numerous luncheon conversations. In the set of notes just referred to, the formula was derived by applying the Hermite-Genocchi integral representation for divided differences, and this approach suggests certain generalizations to nth order divided differences.

The object of the present paper is to publicize this interesting formula, to give an alternate proof of it, and to derive a number of consequences and results related to the alternate approach.

2. Alternate Derivation of the Motzkin-Schoenberg Formula. We begin by recalling several elementary formulas. If z_1 and z_2 are two distinct points in the complex plane, then the equation of the straight line passing through them is

$$(2) \quad \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0,$$

or

$$(2') \qquad \bar{z} = \left(\frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2}\right) z + \left(\frac{z_1 \bar{z}_2 - z_2 \bar{z}_1}{z_1 - z_2}\right).$$

Formula $(2')$ is the simplest example of a "Schwarz reflection function" for an analytic arc, which expresses the equation of the arc in the form

$$(3) \qquad \bar{z} = S(z).$$

(See, e.g., Davis and Pollak [3, p. 6–7, 23–27].)

If z_1, z_2, z_3 are three noncollinear points, then the directed or signed area $A(z_1, z_2, z_3)$ of the triangle with these vertices is given by

$$(4) \qquad A(z_1, z_2, z_3) = \frac{i}{4} \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}.$$

(See, e.g., Deaux [4, p. 59–60].)

Green's Theorem in the plane can, with the help of the Cauchy-Riemann equations, be put in the form

$$(5) \qquad \iint\limits_B f'(z)\, dx\, dy = \frac{i}{2} \int_{\partial B} f\, d\bar{z}; \qquad d\bar{z} = dx - i\, dy.$$

This is a special case of the more general

$$(5') \qquad \iint\limits_B f'(z)\overline{g(z)}\, dx\, dy = \frac{i}{2} \int_{\partial B} f(z)\overline{g(z)}\, d\bar{z}.$$

In formulas (5) and $(5')$ we assume, for simplicity, that the region B has a boundary ∂B that is composed of a finite number of analytic arcs, and that the functions $f(z)$ and $g(z)$ are regular in the closure of B.

Let the vertices of the triangle T in counter-clockwise order be z_1, z_2, z_3. From (5) we obtain

$$(6) \qquad \begin{aligned} \frac{2}{i} \iint\limits_T f''(z)\, dx\, dy &= \int_{\partial T} f'(z)\, d\bar{z} \\ &= \int_{z_1}^{z_2} f'(z)\, d\bar{z} + \int_{z_2}^{z_3} f'(z)\, d\bar{z} + \int_{z_3}^{z_1} f'(z)\, d\bar{z}. \end{aligned}$$

The three contour integrals are taken along the sides of the triangle T_1, T_2, T_3 respectively. Now along T_1 we have from $(2')$,

$$(7) \qquad \bar{z} = A_1 z + B_1, \quad A_1 = \frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2}, \quad B_1 = \frac{z_1 \bar{z}_2 - z_2 \bar{z}_1}{z_1 - z_2}.$$

Therefore along T_1, $d\bar{z} = A_1 dz$. Hence,

$$(8) \qquad (T_1) \int_{z_1}^{z_2} f'(z)\, d\bar{z} = A_1 \int_{z_1}^{z_2} f'(z)\, dz = A_1(f(z_1) - f(z_2)).$$

Similar formulas hold for T_2 and T_3 and therefore,

$$(9) \quad \frac{2}{i} \iint_T f''(z) \, dx \, dy = A_1(f(z_1) - f(z_2)) + A_2(f(z_2) - f(z_3)) + A_3(f(z_3) - f(z_1))$$

$$= (A_3 - A_1) f(z_1) + (A_1 - A_2) f(z_2) + (A_2 - A_3) f(z_3).$$

From (7) and (4),

$$(10) \quad A_3 - A_1 = \frac{\bar{z}_3 - \bar{z}_1}{z_3 - z_1} - \frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2} = \frac{1}{(z_1 - z_2)(z_1 - z_3)} \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$$

$$= \frac{4}{i} \frac{A(z_1, z_2, z_3)}{(z_1 - z_2)(z_1 - z_3)} = \frac{4A}{i(z_1 - z_2)(z_1 - z_3)}.$$

Similar formulas hold for $A_1 - A_2$ and $A_2 - A_3$, and their insertion in (9) leads to (1).

3. Some Applications and Extensions. Write $f(z) \equiv (z - z_1)(z - z_2)(z - z_3)$. Then, $f''(z) = 6z - 2S_1$ where $S_1 = z_1 + z_2 + z_3$. Hence, from (1),

$$\frac{1}{2A} \iint_T (6z - 2S_1) \, dx \, dy = 0.$$

Therefore,

$$\iint_T z \, dx \, dy = \frac{S_1}{3} \iint_T dx \, dy = \frac{S_1 A}{3},$$

or

$$(11) \quad \frac{1}{A} \iint_T z \, dx \, dy = \frac{1}{3}(z_1 + z_2 + z_3),$$

the formula for the center of gravity of a triangle.

By taking the real or imaginary parts of (1), we can obtain the integrals of harmonic functions over triangles. For example, the product of inertia can be obtained in this way. We have

$$\iint_T xy \, dx \, dy = \frac{1}{2} \operatorname{Im} \iint_T z^2 \, dx \, dy = \frac{1}{24} \operatorname{Im} \iint_T (z^4)'' \, dx \, dy$$

$$= \frac{A}{12} \operatorname{Im} \left(\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1^4 & z_2^4 & z_3^4 \end{vmatrix} : \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1^2 & z_2^2 & z_3^2 \end{vmatrix} \right)$$

$$= \frac{A}{12} \operatorname{Im} [(z_1 + z_2 + z_3)^2 - (z_1 z_2 + z_1 z_3 + z_2 z_3)].$$

Hence, if the center of gravity of T is at the origin: $z_1 + z_2 + z_3 = 0$, we obtain the neat formula

(12)   Product of inertia of $T = \iint\limits_{T} xy \, dx \, dy = -\dfrac{A}{12} \operatorname{Im}(z_1 z_2 + z_1 z_3 + z_2 z_3)$.

Formula (1) may be extended to polygons.

LEMMA. *Let* z_1, z_2, $\cdots$, z_n *designate the vertices of a polygon P. Then we can find constants* a_1, $\cdots$, a_n *depending upon* z_1, $\cdots$, z_n *but independent of f, such that for all f regular in the closure of P,*

$$(13) \qquad \iint\limits_{P} f''(z) \, dx \, dy = \sum_{i=1}^{n} a_i f(z_i).$$

If $r \geqq n$, *and* z_{n+1}, z_{n+2}, $\cdots$, z_r *are additional points distinct from* z_1, z_2, $\cdots$, z_n, *and if there are constants* b_1, $\cdots$, b_r *which depend only upon* z_1, $\cdots$, z_r *such that*

$$(14) \qquad \iint\limits_{P} f''(z) \, dx \, dy = \sum_{i=1}^{r} b_i f(z_i)$$

for all f regular in the closure of P, then

$$(15) \qquad b_1 = a_1, \cdots, b_n = a_n, \quad and \quad b_{n+1} = b_{n+2} = \cdots = b_r = 0.$$

Proof. The polygon P may be decomposed into a finite number of triangles T_1, T_2, $\cdots$, T_s such that vertices of these triangles are among the points z_1, $\cdots$, z_n. Write

$$\iint\limits_{P} f'' \, dx \, dy = \sum_{i=1}^{s} \iint\limits_{T_i} f'' \, dx \, dy$$

and apply (1) to each T_i. This proves (13).

From (14) and (13) we have

$$\sum_{i=1}^{n} (a_i - b_i)f(z_i) + \sum_{i=n+1}^{r} b_i f(z_i) = 0,$$

so that with an obvious notation, $\sum_{i=1}^{r} c_i f(z_i) = 0$ for all f regular in the closure of P. In particular, for $f(z) \equiv z^q$, $q = 0, 1, \cdots, r - 1$, we have $\sum_{i=1}^{r} c_i z_i^q = 0$. Since the Vandermonde determinant $\mid z_i^{\,q} \mid \neq 0$, it follows that $c_i = 0$, $i = 1, 2, \cdots, r$ and (15) follows.

What we are asserting in (15) is, essentially, the linear independence of the functionals $L_i(f) = f(z_i)$, $i = 1, 2, \cdots, r$.

We now give an explicit representation of the formula (13).

THEOREM. *With the notation as in the lemma,*

$$\iint\limits_{P} f''(z) \, dx \, dy$$

$$(16) \quad = - \begin{vmatrix} 0 & f(z_1) & f(z_2) & \cdots & f(z_n) \\ \mu_0 & 1 & 1 & & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ \mu_{n-1} & z_1^{\,n-1} & z_n^{\,n-1} & \cdots & z_n^{\,n-1} \end{vmatrix} : \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ \vdots & \vdots & & \vdots \\ z_1^{\,n-1} & z_2^{\,n-1} & \cdots & z_n^{\,n-1} \end{vmatrix},$$

where

(17)
$$\mu_k = \iint_P (z^k)'' \, dx \, dy = k(k-1) \iint_P z^{k-2} \, dx \, dy, \qquad k \geqq 2,$$

$$\mu_0 = \mu_1 = 0.$$

Proof. This comes from inserting $f \equiv 1, f \equiv z, \cdots, f = z^{n-1}$ in (13) and solving the resulting system.

A second representation is the following. For simplicity, we formulate it for a convex polygon.

THEOREM. *Let* $z_1, z_2, \cdots, z_n$ *designate the vertices in counter-clockwise order of a convex polygon* P. *Write* $z_0 = z_n$ *and* $z_{n+1} = z_1$. *Then, for all* $f(z)$ *regular in the closure of* P,

(18)
$$\iint_P f''(z) \, dx \, dy = 2 \sum_{i=1}^{n} \frac{A(z_{i-1}, z_i, z_{i+1})}{(z_i - z_{i-1})(z_i - z_{i+1})} f(z_i).$$

Proof. For fixed i, the polygon P may be decomposed into the triangle T_i whose vertices are z_{i-1}, z_i, and z_{i+1} plus a polygon P_i whose vertices are all the z's save z_i. Hence, by (13) and (1),

(19)
$$\iint_P f''(z) \, dx \, dy = \frac{2A(z_{i-1}, z_i, z_{i+1})}{(z_i - z_{i-1})(z_i - z_{i+1})} f(z_i) + \sum_{j=1; j \neq i}^{n} d_j f(z_j).$$

In view of the lemma, a similar coefficient attaches to each vertex.

COROLLARY.

(20)
$$\sum_{i=1}^{n} \frac{A(z_{i-1}, z_i, z_{i+1})}{(z_i - z_{i-1})(z_i - z_{i+1})} = 0,$$

(21)
$$\sum_{i=1}^{n} \frac{A(z_{i-1}, z_i, z_{i+1})}{(z_i - z_{i-1})(z_i - z_{i+1})} z_i = 0,$$

(22)
$$\sum_{i=1}^{n} \frac{A(z_{i-1}, z_i, z_{i+1})}{(z_i - z_{i-1})(z_i - z_{i+1})} z_i^2 = area \ of \ P.$$

Proof. Insert $f(z) \equiv 1, f(z) \equiv z, f(z) \equiv z^2$ in (18).

COROLLARY. *Let the point* z^* *lie inside the convex polygon* P *whose vertices are* $z_1, z_2, \cdots, z_n$. *Then,*

(23)
$$\sum_{i=1}^{n} \frac{A(z_i, z^*, z_{i+1})}{(z^* - z_i)(z^* - z_{i+1})} = 0.$$

Proof. Decompose P into triangles whose vertices are z_i, z^*, z_{i+1}. In view of (1), $\iint_P f'' \, dx \, dy$ can be expressed as a fixed linear combination of $f(z_1), \cdots, f(z_n)$, $f(z^*)$.

In this linear combination, the coefficient of $f(z^*)$ will be, according to (1),

$$\sum_{i=1}^{n} \frac{2A(z_i, z^*, z_{i+1})}{(z^* - z_i)(z^* - z_{i+1})}.$$

But by (13), $\iint_P f'' \, dx \, dy$ can be expressed as a fixed linear combination of $f(z_1)$,

$\cdots, f(z_n)$. Hence, by the lemma, the coefficient of $f(z^*)$ must vanish and this leads to (23).

It is interesting to notice, that in the case $n = 3$, equation (23) is equivalent to

$$(24) \qquad z^* = z_1 \frac{A(z^*, z_2, z_3)}{A(z_1, z_2, z_3)} + z_2 \frac{A(z_1, z^*, z_3)}{A(z_1, z_2, z_3)} + z_3 \frac{A(z_1, z_2, z^*)}{A(z_1, z_2, z_3)}.$$

The coefficients of z_1, z_2, z_3 in this formulae are the barycentric coordinates of z^* with respect to z_1, z_2, z_3. (See e.g., Deaux [4].)

4. Further Developments. In order to obtain complex formulas for integrals of the type $\iint_T g(x, y) \, dx \, dy$ where g is an analytic function of the two real variables x and y, we can write

$$g(x, y) = g\left(\frac{1}{2}(z + \bar{z}), \frac{1}{2i}(z - \bar{z})\right).$$

Expansions of g in power series now suggest that one should study integrals of the form $\iint_T \bar{z}^n f(z) \, dx \, dy$. We begin with $n = 1$.

THEOREM. Set $z_0 = z_3$ and $z_4 = z_1$. Then if $f(z)$ is regular in the closure of T,

$$
(25) \qquad
\begin{aligned}
\frac{1}{2A} \iint_T \bar{z} f'''(z) \, dx \, dy &= 2 \sum_{i=1}^{3} \frac{\text{Re}\left[(z_i - z_{i-1})(\bar{z}_i - \bar{z}_{i-1})\right]}{(z_i - z_{i-1})^2 (z_i - z_{i+1})^2} f(z_i) \\
&+ \sum_{i=1}^{3} \frac{\bar{z}_i}{(z_i - z_{i-1})(z_i - z_{i+1})} f'(z_i).
\end{aligned}
$$

Proof. Designate the sides of T, properly oriented, by T_1, T_2, T_3. By (5') we have

$$\iint_T \bar{z} f'''(z) \, dx \, dy = \frac{i}{2} \int_{T_1 + T_2 + T_3} \bar{z} f''(z) \, d\bar{z}.$$

Work on T_1 as a prototype:

$$
\begin{aligned}
\frac{i}{2} \int_{T_1} \bar{z} f''(z) \, d\bar{z} &= \frac{i}{2} \int_{z_1}^{z_2} f''(z)(A_1 z + B_1) A_1 \, dz \\
&= \frac{i}{2} \left[[f'(z) A_1 (A_1 z + B_1)]_{z_1}^{z_2} - A_1^2 \int_{z_1}^{z_2} f'(z) \, dz \right] \\
&= \frac{i}{2} [A_1 \bar{z}_2 f'(z_2) - A_1 \bar{z}_1 f'(z_1)] + \frac{i}{2} A_1^2 [f(z_1) - f(z_2)].
\end{aligned}
$$

The first bracket holds since on T_1, $\bar{z} = A_1 z + B$. Hence, $\bar{z}_1 = A_1 z_1 + B_1$, $\bar{z}_2 = A_1 z_2 + B_2$.

Similarly,

$$\frac{i}{2} \int_{T_1} \bar{z} f''(z) \, d\bar{z} = \frac{i}{2} [A_2 \bar{z}_3 f'(z_3) - A_2 \bar{z}_2 f'(z_2)] + \frac{i}{2} A_2^2 [f(z_2) - f(z_3)]$$

and a corresponding expression holds for the integral along T_3. Forming the sum along the three sides, and picking out, for example, the coefficients of $f(z_2)$ and $f'(z_2)$, we have as the contribution from the point z_2:

$$\frac{i}{2} (A_1 - A_2) [\bar{z}_2 f'(z_2) + (A_2 + A_1) f(z_2)].$$

From (10),

$$A_1 - A_2 = \frac{4A}{i(z_2 - z_1)(z_2 - z_3)}.$$

Also,

$$(26) \qquad A_2 + A_1 = \frac{\bar{z}_2 - \bar{z}_3}{z_2 - z_3} + \frac{\bar{z}_1 - \bar{z}_2}{z_1 - z_2} = \frac{2\operatorname{Re}[(z_2 - z_1)(\bar{z}_2 - \bar{z}_3)]}{(z_2 - z_3)(z_2 - z_1)}.$$

Formula (25) now follows.

As an application of this formula, select $f(z) \equiv z(z - z_1)(z - z_2)(z - z_3)$. Then $f'''(z) = 24z - 6S_1$, $S_1 = z_1 + z_2 + z_3$. Now, $f'(z_1) = z_1(z_1 - z_2)(z_1 - z_3)$, etc., hence from (25),

$$(27) \qquad \frac{1}{2A} \iint_T \bar{z}(24z - 6S_1)\, dx\, dy = \bar{z}_1 z_1 + \bar{z}_2 z_2 + \bar{z}_3 z_3.$$

Inserting (11) in (27), we obtain a complex formula for the moment of inertia of a triangle about an axis perpendicular to the triangle and passing through the origin:

$$(28) \qquad \iint_T |z|^2\, dx\, dy = \frac{A}{12}\left[|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_1 + z_2 + z_3|^2\right].$$

If the center of gravity of the triangle is at the origin, we obtain the neat formula

$$(28') \qquad \iint_T |z|^2\, dx\, dy = \frac{A}{12}\left[|z_1|^2 + |z_2|^2 + |z_3|^2\right].$$

The generalization of this theorem is as follows:

THEOREM. *Given a triangle T we can find constants a_{ij} depending upon z_1, z_2, z_3 such that*

$$(29) \qquad \iint_T \bar{z}^m f^{(m+2)}(z)\, dx\, dy = \sum_{i=0}^{m} \sum_{j=1}^{3} a_{ij} f^{(i)}(z_j)$$

for all f regular in the closure of T.

Proof. Use (5) and proceed as before.

Formula (29) may be regarded as a kind of complex three point Taylor expression with exact remainder.

5. Complex Integration over Lunes. Designate the common part of the two circles

$$\left.\begin{aligned} C_1 &: |z - \sigma i|^2 = 1 + \sigma^2 \\ C_2 &: |z + \sigma i|^2 = 1 + \sigma^2 \end{aligned}\right\} \qquad \sigma > 0, \quad i = \sqrt{-1}$$

by L.

These circles both pass through $z = \pm 1$ and their centers are at $z = \sigma i$ and $z = -\sigma i$ respectively. On L_1, the lower boundary of L, we have

$$(30) \qquad \bar{z} = S_1(z) = \frac{1 + \sigma^2}{z - \sigma i} - \sigma i,$$

while on L_2, the upper boundary, we have

$$(30') \qquad \bar{z} = S_2(z) = \frac{1 + \sigma^2}{z + \sigma i} + \sigma i.$$

If $f(z)$ is regular in the closure of L, then by $(5')$ we have

$$(31) \qquad \frac{2}{i} \iint_L f(z) \bar{z}^n \, dx \, dy = \iint_{L_1 + L_2} F(z) \bar{z}^n \, d\bar{z},$$

where $F(z)$ is an indefinite integral of $f(z)$. Now,

$$\int_{L_1} F(z) \bar{z}^n \, d\bar{z} = \int_{L_1} F(z) S_1^n(z) S_1'(z) \, dz$$

$$= F(z) \left. \frac{S_1^{n+1}(z)}{n + 1} \right|_{-1}^{+1} - \frac{1}{n + 1} \int_{L_1} f(z) S_1^{n+1}(z) \, dz.$$

In view of the analyticity of f and S_1, we may take this last integral along the x axis, so that

$$(n + 1) \int_{L_1} F(z) \bar{z}^n \, d\bar{z} = F(1) S_1^{n+1}(1) - F(-1) S_1^{n+1}(-1)$$

$$- \int_{-1}^{+1} f(x) S_1^{n+1}(x) \, dx.$$

Similarly,

$$(n + 1) \int_{L_2} F(z) \bar{z}^n \, d\bar{z} = F(-1) S_2^{n+1}(-1) - F(1) S_2^{n+1}(1)$$

$$- \int_{+1}^{-1} f(x) S_2^{n+1}(x) \, dx.$$

Since $S_1(1) = S_2(1) = 1$, $S_1(-1) = S_2(-1) = -1$,

$$(32) \qquad \frac{2}{i} \iint_L f(z) \bar{z}^n \, dx \, dy = \frac{1}{n + 1} \int_{-1}^{+1} f(x)(S_2^{n+1}(x) - S_1^{n+1}(x)) \, dx.$$

This reduces the complex double integral to a single real integral of the usual sort. The particular values $n = 0$ and $n = 1$ lead to the formulas

$$(33) \qquad \iint_L f(z) \, dx \, dy = \sigma \int_{-1}^{+1} f(x) \frac{1 - x^2}{\sigma^2 + x^2} \, dx,$$

$$(34) \qquad \iint_L \bar{z} f(z) \, dx \, dy = \sigma(1 + \sigma^2) \int_{-1}^{+1} f(x) \frac{x(1 - x^2)}{(x^2 + \sigma^2)^2} \, dx.$$

In a paper that follows, application will be made of equations (1) and (33) to "simple" quadratures.

The author wishes to thank Richard DeMar for a number of useful discussions.

Added in proof. The author wishes to thank Dr. Matthew P. Gaffney for calling my attention to Helmut Grunsky, "Eine Funktionentheoretische Integralformel," *Math. Z.* v. 63, 1955, p. 320–323, where formula (1) will also be found.

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