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Iteration of Triangular Matrices

By Lester J. Senechalle

1. Introduction. In order to calculate scalar functions of a matrix A, it is desirable to have a simple formula for the integral iterates A^n of A. Such a formula was first discovered by Sylvester [1], who expressed A^n as, essentially, a divided difference of the function $f(x) = x^n$. However, Sylvester's formula applies only to the case where the eigenvalues of A are distinct; the case of multiple eigenvalues was subsequently treated by Buchheim [2], and leads to confluent divided differences.

In this paper we give an especially simple formula for A^n when A is an upper triangular matrix. Our algorithm yields only the upper right hand entry of A^n, but this is adequate since every nonzero element of A^n is in fact the upper right-hand entry of the nth iterate of some triangular submatrix of A.

2. Notation. Let $[a_{ij}]$ be an $m \times m$ upper triangular matrix, so that $a_{ij} = 0$ if $i > j$, and for any nonnegative integer n let $[a_{ij}^{(n)}]$ denote the nth iterate of $[a_{ij}]$ under matrix multiplication. The matrix $[a_{ij}^{(n)}]$ is also upper triangular. Moreover, $[a_{ij}^{(0)}] = [\delta_{ij}]$, and $[a_{ij}^{(n+1)}] = [\sum_{k=1}^{m} a_{ik}^{(n)} \cdot a_{kj}]$.

If $(\lambda_1, \cdots, \lambda_k)$ is a chain of complex numbers, then $C(\lambda_1, \cdots, \lambda_k)$ denotes the set of all subchains which have λ_1 as their first element and λ_k as their last element. If $k \geqq 2$ and $i < k$, then $C_i(\lambda_1, \cdots, \lambda_k)$ denotes the set of chains belonging to $C(\lambda_1, \cdots, \lambda_k)$ which have λ_i as their next to last element. Thus $C(\lambda_1, \cdots, \lambda_k) = \bigcup_{i=1}^{k-1} C_i(\lambda_1, \cdots, \lambda_k)$ is a decomposition of $C(\lambda_1, \cdots, \lambda_k)$ into mutually disjoint subsets. For example, $C(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \{(\lambda_1, \lambda_4), (\lambda_1, \lambda_2, \lambda_4), (\lambda_1, \lambda_3, \lambda_4), (\lambda_1, \lambda_2, \lambda_3, \lambda_4)\}$ and $C_3(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \{(\lambda_1, \lambda_3, \lambda_4), (\lambda_1, \lambda_2, \lambda_3, \lambda_4)\}$.

If $\gamma = (\lambda_1, \cdots, \lambda_k)$ is a chain of distinct complex numbers and n is a nonnegative integer, then $q_n(\gamma)$ denotes the divided difference $[\lambda_1 \cdots \lambda_k]$ of the function $f(x) = x^n$ [3, Chapter 1]. Thus

$$q_n(\gamma) = \sum_{i=1}^{k} \frac{\lambda_i^{\,n}}{\prod_{j=1; j \neq i}^{k} (\lambda_i - \lambda_j)}.$$

In particular, $q_n(\gamma) = 0$ for $0 \leqq n < k$, and $q_n(\gamma) = \lambda_1^{\,n}$ if $\gamma = (\lambda_1)$. Furthermore, if $k \geqq 2$, $q_n'(\gamma)$ is defined as $q_n(\gamma')$, where $\gamma' = (\lambda_1, \cdots, \lambda_{k-1})$.

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If $[a_{ij}]$ is an $m \times m$ upper triangular matrix with eigenvalues $\lambda_i = a_{ii}$, $1 \leqq i \leqq m$, and if $\gamma = (\lambda_1 = \lambda_{i_1}, \lambda_{i_2}, \cdots, \lambda_{i_s} = \lambda_m)$, $s \geqq 2$, is in $C(\lambda_1, \cdots, \lambda_m)$, then $\pi(\gamma)$ denotes the product $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{s-1} i_s}$. For example, if

$$m = 6 \quad \text{and} \quad \gamma = (\lambda_1, \lambda_3, \lambda_5, \lambda_6),$$

then $\pi(\gamma) = a_{13}a_{35}a_{56}$. For each one-element chain γ, we define $\pi(\gamma) = 1$.

3. Iteration of Triangular Matrices.

LEMMA. *If $\gamma = (\lambda_1, \cdots, \lambda_k)$, $k \geqq 2$, is a chain of distinct complex numbers and if n is a nonnegative integer, then $q_n'(\gamma) + \lambda_k q_n(\gamma) = q_{n+1}(\gamma)$.*

Proof.

$$q_{n+1}(\gamma) - \lambda_k q_n(\gamma) = \sum_{i=1}^{k-1} \frac{(\lambda_i - \lambda_k)\lambda_i^n}{\prod_{j=1; j \neq i}^{k} (\lambda_i - \lambda_j)} = \sum_{i=1}^{k-1} \frac{\lambda_i^n}{\prod_{j=1; j \neq i}^{k-1} (\lambda_i - \lambda_j)} = q_n'(\gamma).$$

THEOREM. *Let $[a_{ij}]$ be an $m \times m$ upper triangular matrix with distinct eigenvalues $\lambda_i = a_{ii}$, $1 \leqq i \leqq m$. Then for each nonnegative integer n,*

$$a_{1m}^{(n)} = \sum_{\gamma \in C(\lambda_1, \cdots, \lambda_m)} \pi(\gamma) q_n(\gamma).$$

Proof. If $m = 1$, we have $\sum_{\gamma \in C(\lambda_1)} \pi(\gamma) q_n(\gamma) = \lambda_1^n$, which is clearly $a_{11}^{(n)}$.

Suppose that $m > 1$ and that the theorem holds for matrices of order less than m. We show by induction on n that the theorem holds for matrices of order m.

Since $q_0(\gamma) = 0$ for each γ in $C(\lambda_1, \cdots, \lambda_m)$, we have the result for $n = 0$. Now assume the theorem for n. Then

$$\begin{aligned}
a_{1m}^{(n+1)} &= \sum_{j=1}^{m} a_{1j}^{(n)} a_{jm} \\
&= \sum_{j=1}^{m-1} \left(a_{jm} \sum_{\gamma \in C(\lambda_1, \cdots, \lambda_j)} \pi(\gamma) q_n(\gamma) \right) + \lambda_m \sum_{\gamma \in C(\lambda_1, \cdots, \lambda_m)} \pi(\gamma) q_n(\gamma) \\
&= \sum_{j=1}^{m-1} \left(\sum_{\gamma \in C_j(\lambda_1, \cdots, \lambda_m)} \pi(\gamma) q_n'(\gamma) \right) + \lambda_m \sum_{\gamma \in C(\lambda_1, \cdots, \lambda_m)} \pi(\gamma) q_n(\gamma) \\
&= \sum_{\gamma \in C(\lambda_1, \cdots, \lambda_m)} \pi(\gamma)[q_n'(\gamma) + \lambda_m q_n(\gamma)] \\
&= \sum_{\gamma \in C(\lambda_1, \cdots, \lambda_m)} \pi(\gamma) q_{n+1}(\gamma),
\end{aligned}$$

so that the theorem holds for $n + 1$ and the proof is complete.

4. Extensions. The iteration theorem may be extended immediately to the case where the matrix has multiple eigenvalues. We need only regard such matrices as limits of those with distinct eigenvalues, and hence replace $q_n(\gamma)$ in the formula for $a_{1m}^{(n)}$ by a confluent divided difference [3, p. 12–14].

The theorem may also be extended to the case where n is negative and the eigenvalues are nonzero. In fact, the lemma is immediate and only minor alterations are needed in the inductive proof of the theorem.

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On the Evaluation of Certain Determinants

By Jean L. Lavoie

1. Abstract. Using the properties of the generalized Hilbert matrix and familiar results from the theory of hypergeometric series, we evaluate the determinants of certain matrices whose general terms are known explicitly. In certain cases it is even possible to find the analytic expression for the general terms of the inverses.

2. Introduction. In this paper the matrices used are always assumed to be n-square and i and j to be positive integers such that $1 \leqq i, j \leqq n$.

The following elementary properties of determinants will be used:

(I) If K multiplies all the elements in a row (column) of a determinant it multiplies the value of the determinant;

(II) the determinant of a triangular matrix is equal to the product of the n terms along the main diagonal;

(III) if A and B are two matrices then $\det(AB) = \det(A)\det(B)$.

We shall also need Gauss's theorem [1, Theorem 18, p. 49]

$$(1) \qquad {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

and the two following formulas, respectively from [1, example 3, p. 69] and [2, equation 8]:

$$(2) \qquad {}_2F_1\left(\begin{matrix} 1-a, a \\ c \end{matrix} \middle| \frac{1}{2}\right) = \frac{2^{1-c}\Gamma(1/2)\Gamma(c)}{\Gamma\left(\frac{c}{2}+\frac{a}{2}\right)\Gamma\left(\frac{c}{2}-\frac{a}{2}+\frac{1}{2}\right)},$$

and

$$(3) \qquad {}_3F_2\left(\begin{matrix} 1-n, n+p+1, p+j \\ p+j+1, p+1 \end{matrix} \middle| 1\right) = (-1)^{n+1}(p+j)\frac{\Gamma(n)\Gamma(p+1)}{\Gamma(n+p+1)}$$

for j and n positive integers, $1 \leqq j \leqq n$, $p \neq -1, -2, \cdots, -(2n-1)$.

3. Preliminary Results. Let $a_1, a_2, \cdots, a_n$; $b_1, b_2, \cdots, b_n$ be $2n$ distinct but otherwise arbitrary complex numbers. Then it is well known [3, example 3, p. 98] that the determinant of the matrix $H = (h_{ij})$, $h_{ij} = (a_i + b_j)^{-1}$ is

$$\det(H) = \frac{\prod_{r>k}^{1,2,\cdots,n} (a_r - a_k)(b_r - b_k)}{\prod_{i,j}^{1,2,\cdots,n} (a_i + b_j)}.$$

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