Chebyshev Approximations for the Complete Elliptic Integrals K and E

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Abstract. Chebyshev approximations of the Hastings form are given for the complete elliptic integrals K and E. Maximal errors range from 4×10^{-5} down to 4×10^{-18}.

1. Introduction. The complete elliptic integrals are defined by

$$\text{(ia)} \qquad K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1 - k^2 \sin^2 \phi)}} = \frac{\pi}{2}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right), \qquad |k| < 1,$$

and

$$\text{(ib)} \qquad \begin{aligned} E(k) &= \int_0^{\pi/2} \sqrt{(1 - k^2 \sin^2 \phi)}\, d\phi, \qquad |k| \leqq 1, \\ &= \frac{\pi}{2}\, {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; k^2\right), \qquad |k| < 1, \end{aligned}$$

where ${}_2F_1(a, b; c; z)$ is Gauss' hypergeometric series [1]. Another useful form, a modified Legendre form [2], can be obtained from (i) by means of standard transformations on the hypergeometric series [3]. Thus

$$\text{(iia)} \qquad K(k) = K_1(\eta) + K_2(\eta) \ln (1/\eta), \qquad 0 \leqq \eta < 1,$$

where

$$K_1(\eta) = \ln 4 + \sum_{n=1}^{\infty} \frac{1^2 \cdot 3^2 \cdots (2n-1)^2}{2^2 \cdot 4^2 \cdots (2n)^2} \left[\ln 4 - 2 \sum_{j=1}^{2n} \frac{(-1)^{j-1}}{j}\right] \eta^n,$$

$$K_2(\eta) = \frac{1}{2}\left[1 + \sum_{n=1}^{\infty} \frac{1^2 \cdot 3^2 \cdots (2n-1)^2}{2^2 \cdot 4^2 \cdots (2n)^2} \eta^n\right],$$

and η is the square of the complementary modulus,

$$\eta = 1 - k^2 = k'^2.$$

Similarly,

$$\text{(iib)} \qquad E(k) = E_1(\eta) + E_2(\eta) \ln \frac{1}{\eta}, \qquad 0 \leqq \eta < 1,$$

where

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$$E_1(\eta) = 1 + \frac{\eta}{2}[\ln 4 - 1] + \sum_{n=2}^{\infty} \frac{1^2 \cdot 3^2 \cdots (2n-3)^2(2n-1)}{2^2 \cdot 4^2 \cdots (2n-2)^2(2n)} \left[\ln 4 - \frac{2}{1} + \frac{2}{2} - \cdots - \frac{2}{2n-3} + \frac{2}{2n-2} - \frac{1}{2n-1} + \frac{1}{2n} \right] \eta^n$$

and

$$E_2(\eta) = \frac{\eta}{4} + \frac{1}{2} \sum_{n=2}^{\infty} \frac{1^2 \cdot 3^2 \cdots (2n-3)^2(2n-1)}{2^2 \cdot 4^2 \cdots (2n-2)^2(2n)} \eta^n.$$

From (ia) and (ib) it is apparent that

$$K(0) = E(0) = \pi/2,$$

while (iia) and (iib) show that

$$K(k) = \ln(4/k') + O(k'^2 \ln k'), \qquad k' \to 0$$

(i.e., $K(k)$ becomes logarithmically infinite as $k \to 1$), and

$$E(1) = 1.$$

2. Approximating Forms. For a given function one approximating form is said to be more efficient than another if, for a given number of coefficients, the error of approximation is less. The more efficient approximation forms generally contain much of the analytic behavior of the function being approximated. To be useful, a form should also be simple. While rational functions, ratios of polynomials, are simple and generally more efficient than polynomials, neither form is particularly efficient in approximating $K(k)$ because of the logarithmic behavior as $k \to 1$. The form

$$K^*(k) = \ln\left(\frac{1}{\eta}\right) + R(\eta),$$

where $R(\eta)$ is a rational function, incorporates this logarithmic behavior and is more efficient than pure rational functions. The form

$$K^*(k) = \ln\left(\frac{1}{\eta}\right) R(\eta)$$

also contains the behavior of the first derivative and is thus even more efficient, as our experiments have verified. The most efficient form involving rational functions is probably

$$K^*(k) = R_1(\eta) + R_2(\eta) \ln\left(\frac{1}{\eta}\right).$$

For practical reasons we were restricted to trying this form with $R_1(\eta)$ and $R_2(\eta)$ pure polynomials in η. Additional analytic behavior can be built into this form by requiring

$$R_1(0) = \ln 4, \qquad R_2(0) = 1/2, \qquad \text{and} \qquad R_1(1) = \pi/2.$$

Thus, the final approximation form is one first used by Hastings [4],

(iiia) $$K^*(k) = \sum_{i=0}^{n} a_i \eta^i + \ln\left(\frac{1}{\eta}\right) \sum_{j=0}^{m} b_j \eta^j,$$

where

$$a_0 = \ln 4,$$

$$\sum_{i=0}^{n} a_i = \pi/2,$$

and

$$b_0 = 1/2.$$

By similar reasoning we were led to the Hastings form for $E(k)$,

(iiib) $$E^*(k) = \sum_{i=0}^{n} c_i \eta^i + \ln\left(\frac{1}{\eta}\right) \sum_{j=0}^{m} d_j \eta^j,$$

where

$$c_0 = 1, \qquad \sum_{i=0}^{n} c_i = \pi/2, \qquad \text{and} \qquad d_0 = 0.$$

TABLE Ia

Maximum approximation errors for $n = m$

n	$\max \mid \delta_K \mid$	$\max \mid \delta_E \mid$
2	2.99 (−05)	3.91 (−05)
3	6.02 (−07)	7.32 (−07)
4	1.34 (−08)	1.57 (−08)
5	3.19 (−10)	3.62 (−10)
6	7.85 (−12)	8.74 (−12)
7	1.99 (−13)	2.18 (−13)
8	5.12 (−15)	5.56 (−15)
9	1.34 (−16)	1.45 (−16)
10	3.56 (−18)	3.81 (−18)

TABLE Ib

Maximum approximation errors for $n \neq m$

n	m	$\max \mid \delta_K \mid$	$\max \mid \delta_E \mid$
1	3	2.81 (−04)	4.13 (−04)
3	1	8.20 (−05)	1.14 (−04)
2	4	2.53 (−06)	3.19 (−06)
4	2	1.26 (−06)	1.57 (−06)
3	5	3.85 (−08)	4.56 (−08)
5	3	2.41 (−08)	2.84 (−08)
4	6	7.29 (−10)	8.35 (−10)
6	4	5.17 (−10)	5.90 (−10)

TABLE II

Coefficients for $K^* = \ln 4 + \sum_{i=1}^{n} a_i \eta^i + \ln\left(\frac{1}{\eta}\right)\left[\frac{1}{2} + \sum_{i=1}^{n} b_i \eta^i\right]$

$\ln 4 = 1.38629\ 43611\ 19890\ 61883$

i	a_i		b_i	
		$n = 2$		
1	1.119697	(−01)	1.213486	(−01)
2	7.253230	(−02)	2.887472	(−02)
		$n = 3$		
1	9.79324 618	(−02)	1.24750 843	(−01)
2	5.45433 073	(−02)	6.01197 058	(−02)
3	3.20261 966	(−02)	1.09455 763	(−02)
		$n = 4$		
1	9.66633 8350	(−02)	1.24985 9468	(−01)
2	3.58998 0090	(−02)	6.88029 5505	(−02)
3	3.74253 9571	(−02)	3.32852 1016	(−02)
4	1.45133 8556	(−02)	4.41839 8230	(−03)
		$n = 5$		
1	9.65786 19622 6	(−02)	1.24999 29597 5	(−01)
2	3.15594 31627 5	(−02)	7.01487 57782 9	(−02)
3	2.37612 24857 6	(−02)	4.49838 75539 9	(−02)
4	2.59628 88452 6	(−02)	1.87516 60276 9	(−02)
5	6.63980 11146 8	(−03)	1.84723 41632 3	(−03)
		$n = 6$		
1	9.65738 43102 223	(−02)	1.24999 96748 737	(−01)
2	3.09539 55531 153	(−02)	7.02980 97586 169	(−02)
3	1.69419 59131 641	(−02)	4.81598 84398 615	(−02)
4	1.97429 05159 930	(−02)	3.07244 21769 603	(−02)
5	1.72371 88608 291	(−02)	1.05049 11494 346	(−02)
6	3.05211 41417 676	(−03)	7.89679 91858 043	(−04)
		$n = 7$		
1	9.65736 02051 6771	(−02)	1.24999 99858 5309	(−01)
2	3.08909 63386 1795	(−02)	7.03114 10585 3296	(−02)
3	1.52618 32062 2534	(−02)	4.87379 51094 5218	(−02)
4	1.25565 69354 3211	(−02)	3.57218 44300 7327	(−02)
5	1.68695 68596 7517	(−02)	2.09857 67733 6790	(−02)
6	1.09423 81068 8623	(−02)	5.81807 96187 1996	(−03)
7	1.40704 91549 6101	(−03)	3.42805 71922 9748	(−04)

TABLE II—*Continued*

i	a_i		b_i	
		$n = 8$		
1	9.65735 90797 58901 8	(−02)	1.24999 99994 11792 3	(−01)
2	3.08855 73486 75269 4	(−02)	7.03124 26464 62736 1	(−02)
3	1.49789 88178 70462 9	(−02)	4.88180 58565 40395 2	(−02)
4	9.65875 79861 75311 3	(−03)	3.70683 98934 15542 2	(−02)
5	1.12089 18554 64409 2	(−02)	2.71898 61116 78825 0	(−02)
6	1.38556 01247 15656 0	(−02)	1.41053 80776 15804 8	(−02)
7	6.69055 09906 89793 6	(−03)	3.18313 09927 86288 6	(−03)
8	6.49984 43329 39018 0	(−04)	1.50491 81783 60188 3	(−04)
		$n = 9$		
1	9.65735 90301 74252 85	(−02)	1.24999 99999 76406 58	(−01)
2	3.08851 73001 89970 99	(−02)	7.03124 95459 54660 82	(−02)
3	1.49420 29142 28207 83	(−02)	4.88271 55048 11800 99	(−02)
4	8.92664 62945 56466 20	(−03)	3.73355 46682 28602 96	(−02)
5	7.51938 67218 08381 02	(−03)	2.95037 29348 68871 30	(−02)
6	1.05899 53620 98935 85	(−02)	2.06902 40005 10084 04	(−02)
7	1.07959 90490 59163 49	(−02)	9.28116 03829 68604 19	(−03)
8	3.96847 09020 98978 19	(−03)	1.72161 47097 98652 12	(−03)
9	3.00725 19903 68648 38	(−04)	6.66317 52464 60731 51	(−05)
		$n = 10$		
1	9.65735 90280 85625 5384	(−02)	1.24999 99999 99080 8051	(−01)
2	3.08851 46271 30518 9866	(−02)	7.03124 99739 03835 2054	(−02)
3	1.49380 13532 68716 5242	(−02)	4.88280 41906 86239 7978	(−02)
4	8.78980 18745 55064 6778	(−03)	3.73777 39758 62360 4144	(−02)
5	6.17962 74460 53317 6084	(−03)	3.01248 49012 89893 0266	(−02)
6	6.84790 92826 24505 1197	(−03)	2.39319 13323 11079 0077	(−02)
7	9.84892 93221 76893 7682	(−03)	1.55309 41631 97720 3877	(−02)
8	8.00300 39806 49985 3708	(−03)	5.97390 42991 55429 1551	(−03)
9	2.29663 48983 96958 6869	(−03)	9.21554 63496 32498 4638	(−04)
10	1.39308 78570 06646 7279	(−04)	2.97002 80966 55561 2066	(−05)

The high efficiency of these forms might be expected from their similarity to the modified Legendre forms (iia) and (iib).

3. Computations. The Remes algorithm for computing rational Chebyshev approximations [5] was programmed in 25-decimal floating point arithmetic on a CDC 3600. The functions $K(k)$ and $E(k)$ were computed as needed using the standard Gauss arithmetic-geometric mean process [6]. Because of the nature of the approximating forms, the error curves

$$\delta_K(k) = K(k) - K^*(k)$$

and

$$\delta_E(k) = E(k) - E^*(k)$$

TABLE III

Coefficients for $E^* = 1 + \sum_{i=1}^{n} c_i\eta^i + \ln\left(\frac{1}{\eta}\right)\sum_{i=1}^{n} d_i\eta^i$

i	c_i		d_i	
	$n = 2$			
1	4.630106	(−01)	2.452740	(−01)
2	1.077857	(−01)	4.125321	(−02)
	$n = 3$			
1	4.44789 300	(−01)	2.49698 607	(−01)
2	8.50922 292	(−02)	8.15096 894	(−02)
3	4.09147 972	(−02)	1.38343 651	(−02)
	$n = 4$			
1	4.43251 5145	(−01)	2.49983 6641	(−01)
2	6.26076 1942	(−02)	9.20010 9374	(−02)
3	4.75740 4429	(−02)	4.06946 8414	(−02)
4	1.73631 4854	(−02)	5.26378 9328	(−03)
	$n = 5$			
1	4.43152 87472 6	(−01)	2.49999 20273 6	(−01)
2	5.75669 98484 1	(−02)	9.35649 07830 7	(−02)
3	3.17611 45524 7	(−02)	5.42605 24448 7	(−02)
4	3.06623 47457 2	(−02)	2.18360 21169 3	(−02)
5	7.65296 06032 8	(−03)	2.12479 18284 5	(−03)
	$n = 6$			
1	4.43147 46159 513	(−01)	2.49999 96385 465	(−01)
2	5.68815 88013 808	(−02)	9.37340 05947 003	(−02)
3	2.40523 63568 173	(−02)	5.78528 08337 762	(−02)
4	2.36579 46984 506	(−02)	3.53596 51640 905	(−02)
5	1.96232 72084 535	(−02)	1.18851 15619 289	(−02)
6	3.43369 45487 476	(−03)	8.87417 84464 644	(−04)
	$n = 7$			
1	4.43147 19346 7733	(−01)	2.49999 99844 8655	(−01)
2	5.68115 68105 3803	(−02)	9.37488 06209 8189	(−02)
3	2.21862 20699 3846	(−02)	5.84950 29706 6166	(−02)
4	1.56847 70023 9786	(−02)	4.09074 82159 3164	(−02)
5	1.92284 38902 2977	(−02)	2.35091 60256 4984	(−02)
6	1.21819 48148 6695	(−02)	6.45682 24731 5060	(−03)
7	1.55618 74474 5296	(−03)	3.78886 48734 9367	(−04)

TABLE III—*Continued*

i	c_i		d_i	
		$n = 8$		
1	4.43147 18112 15580 6	(−01)	2.49999 99993 61762 2	(−01)
2	5.68056 57874 69535 8	(−02)	9.37499 20249 68011 3	(−02)
3	2.18762 20647 18619 8	(−02)	5.85828 39536 55902 4	(−02)
4	1.25105 92410 84464 4	(−02)	4.23828 07456 94790 0	(−02)
5	1.30341 46073 73143 2	(−02)	3.03027 47728 41284 8	(−02)
6	1.53771 02528 55201 9	(−02)	1.55251 29948 04072 1	(−02)
7	7.33561 74974 29036 5	(−03)	3.48386 79435 89649 2	(−03)
8	7.09809 64089 98722 9	(−04)	1.64272 10797 04802 5	(−04)
		$n = 9$		
1	4.43147 18058 33681 37	(−01)	2.49999 99999 74614 23	(−01)
2	5.68052 23329 30828 95	(−02)	9.37499 95116 36706 73	(−02)
3	2.18361 31405 48689 67	(−02)	5.85927 07184 26527 39	(−02)
4	1.17167 66944 65772 28	(−02)	4.26725 10126 59175 23	(−02)
5	9.03552 77375 40881 84	(−03)	3.28110 69172 72106 18	(−02)
6	1.18419 25995 50124 94	(−02)	2.26603 09891 60412 21	(−02)
7	1.17858 41008 73393 55	(−02)	1.00879 58494 37510 04	(−02)
8	4.30253 77747 93116 59	(−03)	1.86453 79184 06336 32	(−03)
9	3.25192 01550 63904 18	(−04)	7.20316 96345 71545 99	(−05)
		$n = 10$		
1	4.43147 18056 08895 2648	(−01)	2.49999 99999 99017 7208	(−01)
2	5.68051 94567 55915 6648	(−02)	9.37499 99721 20314 0658	(−02)
3	2.18318 11676 13048 1568	(−02)	5.85936 61255 53149 1732	(−02)
4	1.15695 95745 29540 2175	(−02)	4.27178 90547 38309 5644	(−02)
5	7.59509 34225 59432 2802	(−03)	3.34789 43665 76162 6232	(−02)
6	7.82040 40609 59554 1727	(−03)	2.61450 14700 31387 8932	(−02)
7	1.07706 35039 86645 5473	(−02)	1.68040 23346 36338 4981	(−02)
8	8.63844 21736 04074 4302	(−03)	6.43214 65864 38301 7666	(−03)
9	2.46850 33304 60722 7339	(−03)	9.89833 28462 25384 7867	(−04)
10	1.49466 21757 18132 6771	(−04)	3.18591 95655 50157 1800	(−05)

vanished for $k = 0$ and $k = 1$. The Remes algorithm thus did not require calculations of K or E close enough to $k = 0$ or $k = 1$ to give any numerical difficulties.

All error curves were levelled to at least four significant figures in the maximal error. As a final check each approximation was separately tested for 2000 pseudo-random arguments against a 25-decimal routine based on the Gauss process. The maximal errors in these tests corresponded in magnitude and location with those given by the error curves in the Remes algorithm.

4. Results. Although approximations of many different forms were computed, only those of form (iii) with $n = m$ are reported here. Table I lists the maximal errors, including a few representative cases with $n \neq m$ to show $n = m$ is most efficient. Tables II and III list the corresponding coefficients to an accuracy slightly greater than that justified by the maximal errors. Reasonable rounding should not seriously affect the maximal errors.

The cases $n = 2, 3$, and 4 were first given by Hastings [4] and are reported here only to show the agreement between his calculations and ours.

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