M-function and the appropriate asymptotic series as above. At and above $r = 3.3$, the asymptotic series agrees to eight significant figures with the result from the M-function. The results are reported in Table IV. Values at $n = 0$ have been omitted, for they are simply the error function

$$T\left(1, \frac{1}{2}, r\right) = 2\pi^{-1/2} \int_0^r e^{-y^2}\, dy. \tag{5}$$

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Tables of Values of σ_{2s} Relating to Weierstrass' Elliptic Function

By Chih-Bing Ling

The coefficient σ_{2s} is defined by the double series

$$\sigma_{2s} = \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(m + n\omega)^{2s}} \qquad (s \geqq 2), \tag{1}$$

where the prime on the summation sign indicates the omission of simultaneous zeros of m and n; ω being a complex quantity. Such a coefficient occurs in the expansion of Weierstrass' elliptic function. Besides, it occurs also in the expansions of Weierstrass' Sigma and Zeta functions.

In two previous papers [1], [2], the author and his associate evaluated to 16D the Weierstrass' elliptic function at half periods and also the two coefficients σ_4 and σ_6 for the following two cases, namely: (i) when $\omega = ai$, and (ii) when $\omega = \frac{1}{2} + ci$. In the former case the primitive period-parallelogram is a rectangle and in the latter a rhombus. It appears that these two cases are the only cases in which σ_4 and σ_6 are both real.

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TABLE 1

σ_{2s} *for rectangular primitive period-parallelogram* $\omega = ai$

2s	$a = 1$	$a = 1.25$	$a = 1.5$
4	3.15121 20021 53898	2.36702 93923 35617	2.20660 15468 91272
6	0	1.63147 62559 94511	1.95170 97194 76020
8	4.25577 30353 65190	2.40121 20617 91737	2.08675 30228 89838
10	0	1.75534 19321 98488	1.95756 62209 44780
12	3.93884 90128 27970	2.13468 48216 88842	2.01834 84864 02495
14	0	1.91766 73602 06007	1.99364 70698 21010
16	4.01569 50330 25025	2.05589 62440 31736	2.00275 30857 22200
18	0	1.96321 17063 50306	1.99869 40518 66112
20	3.99609 67531 76290	2.02325 66800 05057	2.00062 44668 48807

2s	$a = 1.75$	$a = 2$	$a = 2.5$
4	2.17336 30560 32070	2.16645 82514 80805	2.16472 47593 59381
6	2.01747 33403 07279	2.03110 95062 61006	2.03453 15813 20540
8	2.02436 01314 25025	2.01151 77237 46828	2.00829 99787 64371
10	1.99304 63747 06116	2.00014 27043 18299	2.00190 94944 46894
12	2.00378 38243 67042	2.00115 83973 86476	2.00052 07677 71726
14	1.99920 61660 36118	1.99995 03073 42851	2.00011 52532 11914
16	2.00026 57862 50650	2.00006 56367 94584	2.00003 19311 96390
18	1.99993 65470 76706	2.00000 09718 89557	2.00000 74333 00538
20	2.00002 75856 67379	2.00000 34066 80746	2.00000 19325 10051

2s	$a = 3$	$a = 3.5$	$a = 4$
4	2.16464 98507 19257	2.16464 66136 27787	2.16164 64737 40389
6	2.03467 94456 07301	2.03468 58353 70785	2.03468 61114 97443
8	2.00816 09898 08100	2.00815 49836 67249	2.00815 47241 18587
10	2.00198 57082 70629	2.00198 90015 14453	2.00198 91438 27948
12	2.00049 34084 29168	2.00049 22264 89062	2.00049 21754 13479
14	2.00012 21836 38509	2.00012 24827 60770	2.00012 24956 86327
16	2.00003 06233 00610	2.00003 05670 58490	2.00003 05646 28566
18	2.00000 76260 49407	2.00000 76342 17909	2.00000 76345 70601
20	2.00000 19089 12063	2.00000 19079 66624	2.00000 19079 25907

2s	$a = 5$	$a = 6$	$a = \infty$
4	2.16464 64674 34075	2.16464 64674 22298	2.16464 64674 22276
6	2.03468 61239 45609	2.03468 61239 68855	2.03468 61239 68898
8	2.00815 47124 17780	2.00815 47123 95930	2.00815 47123 95889
10	2.00198 91502 43633	2.00198 91502 55614	2.00198 91502 55636
12	2.00049 21731 10924	2.00049 21731 06624	2.00049 21731 06616
14	2.00012 24962 69027	2.00012 24962 70115	2.00012 24962 70117
16	2.00003 05645 19022	2.00003 05645 18818	2.00003 05645 18817
18	2.00000 76345 86500	2.00000 76345 86530	2.00000 76345 86530
20	2.00000 19079 24071	2.00000 19079 24068	2.00000 19079 24068

TABLE 2

σ_{2s} for rhombic primitive period-parallelogram $\omega = \frac{1}{2} + ci$

$2s$	$c = 0.05$	$c = 0.1$	$c = 0.15$
8	2.00815 47123 73997 (8)	7.84378 69206 07052 (5)	3.01929 16328 50742 (4)
10	−2.00198 91502 67640 (10)	−1.95514 53063 95460 (7)	−3.41536 56800 44365 (5)
12	2.00049 21731 02308 (12)	4.88394 43264 04845 (8)	3.75475 62384 31253 (6)
14	−2.00012 24962 71208 (14)	−1.22078 23006 98475 (10)	−4.18394 56319 30360 (7)
16	2.00003 05645 18612 (16)	3.05180 23853 65539 (11)	4.64613 60590 62755 (8)
18	−2.00000 76345 86560 (18)	−7.62942 43929 84776 (12)	−5.16211 95328 00326 (9)
20	2.00000 19079 24064 (20)	1.90735 04323 37422 (14)	5.73610 81251 78303 (10)

$2s$	$c = 0.2$	$c = 0.25$	$c = (\sqrt{3})/6$
8	2.52163 09821 53529 (3)	1.60816 72114 13027 (2)	0
10	−2.06652 59559 96521 (4)	−2.16016 01031 74883 (3)	0
12	1.18224 64731 33443 (5)	1.14373 13979 75789 (4)	4.38102 75393 67609 (3)
14	−7.31607 11619 11153 (5)	−1.93129 15172 21652 (4)	0
16	4.73515 63962 10678 (6)	1.49613 81950 93459 (5)	0
18	−2.88715 41916 22228 (7)	−5.90720 55456 25546 (5)	−1.18092 45640 84411 (5)
20	1.82128 81085 50568 (8)	1.65199 83685 00529 (6)	0

$2s$	$c = 0.3$	$c = 0.35$	$c = 0.4$
8	5.63025 42542 23542	7.16930 75248 29846 (1)	1.03390 63045 23309 (2)
10	2.32012 66344 74148 (2)	4.80990 66906 49199 (2)	2.92534 57548 99078 (2)
12	3.46111 87050 18462 (3)	8.97879 25023 15439 (2)	−1.71498 75922 00195 (2)
14	−3.88125 32763 35660 (2)	−2.87125 35619 33968 (3)	−2.09707 55966 21308 (3)
16	−8.86356 42829 63524 (3)	−9.54423 92691 97711 (3)	−9.20745 08676 60191 (2)
18	−8.25716 36191 04954 (4)	−1.29877 83006 28397 (3)	1.10420 93191 62505 (4)
20	1.86446 03449 29178 (4)	7.55114 32790 14949 (4)	1.80282 01385 45422 (4)

$2s$	$c = 0.45$	$c = 0.5$	$c = 0.6$
8	9.38182 15108 54620 (1)	6.80923 68565 84303 (1)	2.40761 83490 49172 (1)
10	1.02247 10331 45155 (2)	0	−3.56686 32971 02146 (1)
12	−3.67282 32707 42029 (2)	−2.52086 33682 09901 (2)	−3.38381 33928 98927 (1)
14	−6.98217 75294 48321 (2)	0	1.23389 03791 12239 (2)
16	1.52950 49149 89117 (3)	1.02801 79284 54406 (3)	2.71377 90638 05890 (1)
18	4.06917 86029 98089 (3)	0	−3.39489 41567 49287 (2)
20	−5.49558 19839 58451 (3)	−4.09200 30752 52521 (3)	1.36649 36657 00840 (2)

$2s$	$c = 0.7$	$c = 0.8$	$c = (\sqrt{3})/2$
8	5.38240 25668 63314	4.69672 52853 83047 (−1)	0
10	−1.60597 04560 47814 (1)	−3.50938 09680 31528	0
12	1.17745 72679 96845 (1)	9.36466 92220 97731	6.00963 99716 97680
14	2.62676 92932 11863 (1)	1.69560 38211 21116	0
16	−3.70457 93339 91385 (1)	−6.97787 62553 99495	0
18	−8.11071 55876 53775	1.12245 26806 10248 (1)	5.99971 83563 70526
20	8.21817 27661 07769 (1)	4.25137 66179 49461	0

$2s$	$c = 0.9$	$c = 1$	$c = 1.25$
8	6.95305 88961 42480 (−2)	6.28190 31695 82138 (−1)	1.65257 60804 64137
10	9.55749 43667 38949 (−1)	2.10953 13507 56722	2.16690 95208 35008
12	4.77236 80355 54756	2.79231 29833 39329	1.98348 04444 99797
14	1.77675 80651 60856 (−1)	1.17876 67951 79231	1.96389 28984 76070
16	1.35561 68245 22299	2.28292 57127 28056	2.03373 41863 17008
18	4.26373 75865 78348	2.25342 00842 38261	1.98403 72530 44286
20	3.16659 30321 50863 (−1)	1.57546 84147 83982	2.00252 74528 81507

TABLE 2—*Continued*

2s	c = 1.5	c = 1.75	c = 2
8	1.93117 59742 41995	1.99201 92813 77654	2.00479 4[illegible]255 01028
10	2.04288 06029 72341	2.01077 93173 51130	2.00382 90013 84623
12	1.98769 82115 96826	1.99741 92831 10871	1.99983 54023 52954
14	2.00147 51412 17272	2.00081 74623 42361	2.00028 51192 05692
16	2.00116 04844 15531	1.99996 17878 77766	2.00000 26849 24136
18	1.99908 32446 34909	1.99998 20894 46474	2.00001 01215 55444
20	2.00041 53393 29970	2.00002 08536 24298	2.00000 23372 49041
2s	**c = 2.5**	**c = 3**	**c = 4**
8	2.00800 94516 75405	2.00814 84349 94225	2.00815 47006 73191
10	2.00206 87937 48951	2.00199 25922 17645	2.00198 91566 83325
12	2.00046 35960 95424	2.00049 09378 17032	2.00049 21707 99753
14	2.00012 97214 64381	2.00012 28088 68366	2.00012 24968 53908
16	2.00002 92112 73160	2.00003 05057 62108	2.00003 05644 09069
18	2.00000 78280 74601	2.00000 76431 09091	2.00000 76346 02459
20	2.00000 18869 38644	2.00000 19069 42796	2.00000 19079 22229
2s	**c = 5**	**c = 6**	**c = ∞**
8	2.00815 47123 73997	2.00815 47123 95848	2.00815 47123 95889
10	2.00198 91502 67640	2.00198 91502 55659	2.00198 91502 55636
12	2.00049 21731 02308	2.00049 21731 06608	2.00049 21731 06616
14	2.00012 24962 71208	2.00012 24962 70119	2.00012 24962 70117
16	2.00003 05645 18612	2.00003 05645 18817	2.00003 05645 18817
18	2.00000 76345 86560	2.00000 76345 86530	2.00000 76345 86530
20	2.00000 19079 24064	2.00000 19079 24068	2.00000 19079 24068

N.B. (*n*) at the end of the number stands for the factor 10^n.

In the present note, the coefficient σ_{2s} in the preceding two cases will be evaluated also to 16D. The following recurrence relation is used in the evaluation:

$$\tfrac{1}{3}(s-3)\,(2s+1)C_{2s} = C_4C_{2s-4} + C_6C_{2s-6} + C_8C_{2s-8} + \cdots + C_{2s-4}C_4 \qquad (s = 4, 5, 6, \cdots), \tag{2}$$

where

$$C_{2s} = (2s-1)\sigma_{2s} \qquad (s \geqq 2) \tag{3}$$

so that σ_{2s} for $s \geqq 4$ can be computed from the two initial coefficients σ_4 and σ_6. This relation is obtained by equating the coefficients of the term z^{2n-2} from both sides of the expansion of the equation [3]

$$\frac{d^2}{dz^2}\wp(z) = 6\{\wp^2(z) + 5\sigma_4\}, \tag{4}$$

where $\wp(z)$ is Weierstrass' elliptic function of double periods 1 and ω.

It is interesting to note the following conversion formulas. By expressing σ_{2s} as a function of a or c, respectively, we have (i) in the case of rectangular primitive period-parallelogram

$$\sigma_{2s}(a') = (-1)^s a^{2s}\sigma_{2s}(a) \qquad (aa' = 1) \tag{5}$$

and (ii) in the case of rhombic primitive period-parallelogram

$$\sigma_{2s}(c') = (-1)^s(2c)^{2s}\sigma_{2s}(c) \qquad (cc' = \tfrac{1}{4}). \tag{6}$$

The computation has been carried out up to $2s = 50$ with adequate guarding figures provided for σ_4 and σ_6. The values are then rounded off to 16D. Individual check is made on the last two coefficients by direct summation of the double series. The results up to $2s = 20$ are shown in Tables 1 and 2. In Table 2, the values of σ_4 and σ_6 are not included, which may be found in reference 2. The complete table is deposited in the UMT file in the office of the journal.

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A Method for the Computation of the Error Function of a Complex Variable

By Otto Neall Strand

Abstract. This paper presents a method of computing $\text{erf } z \equiv (2/\sqrt{\pi}) \int_0^z e^{-u^2}\, du$, where z is complex. It is shown that $\text{erfc } z \equiv 1 - \text{erf } z$ has no zeros in the right-hand half plane. An estimate of $|\text{erfc } z|$ is derived.

The error function of a complex variable, denoted by erf z, is defined by the equation $\text{erf } z = (2/\sqrt{\pi})\int_0^z e^{-u^2}\, du$, where z is complex. This function arises in many problems of physics and engineering. Several methods [1], [2], [3] have been devised for the computation of erf z and closely-related functions, and several tabulations [4], [5], [6] have been made. The method to be described below has two features which make it relatively simple to use: (1) the phase enters in a simple explicit manner; and (2) the major portion of the computation consists of the accumulation of two series of positive terms for which each term (after the first) may be calculated by a simple recursion without the use of transcendental functions. For the particular FORTRAN double-precision programs which were written for comparison, the average computing time for the method of this paper was found to be approximately $\frac{7}{10}$ of that for Salzer's first method [7] for an equally-spaced grid of points throughout the region defined by $0 < |z| < 6.6$ and $0 \leqq \arg z < \pi/2$. The relative difference between results from the two methods was less than 10^{-13} throughout this region.

Since the relations $\text{erf }(-z_0) = -\text{erf } z_0$ and $\overline{\text{erf }(z_0)} = \text{erf }(\bar{z}_0)$ may always be employed to reduce the computation to one involving z_0 in the first quadrant, the

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