A New Integral Representation of the Bessel Coefficients

By P. Razelos

The modified Bessel coefficients I_n are defined by the series [1]

$$I_n(t) = \sum_{q=0}^{\infty} \frac{\left(\frac{t}{2}\right)^{2q+n}}{q!(q+n)!}, \tag{1}$$

with the integral representation [1]

$$I_n(t) = \frac{1}{\pi} \int_0^{\pi} e^{t \cos x} \cos nx \, dx. \tag{2}$$

An integral representation of the coefficients $I_n(t)$ is presented here where the path of integration is extended to infinity.

Theorem: *The integral*

$$A_n(t) = \frac{2}{\pi} \int_0^{\infty} e^{t \cos x} \cos nx \frac{\sin \epsilon x}{x} \, dx$$

is the modified coefficient $I_n(t)$ for any $0 < \epsilon < 1$. There are many ways by which the theorem can be proved, but we give here the following proof which consists of a straightforward evaluation of the integral A_n. The function $e^{t \cos x}$ is expanded in a power series (which is absolutely convergent for all t) and we then integrate term by term.

$$e^{t \cos x} = \sum_{r=0}^{\infty} \frac{(t \cos x)^r}{r!}. \tag{3}$$

Let us define

$$Q_n^r = \frac{2}{\pi} \int_0^{\infty} \cos^r x \cos nx \frac{\sin \epsilon x}{x} \, dx. \tag{4}$$

Then

$$A_n(t) = \sum_{r=0}^{\infty} \frac{Q_n^r}{r!} t^r. \tag{5}$$

Introducing the expansion

$$\cos^r x = \frac{1}{2^{r-1}} \sum_{k=0}^{(r/2)-\delta} \binom{r}{k} \cos(r-2k)x \tag{6}$$

(where $\delta = 1$ or $\frac{1}{2}$ for r even or odd, respectively) into (4), we obtain

$$Q_n^r = \frac{1}{2^r} \sum_{k=0}^{(r/2)-\delta} \binom{r}{k} [g(r-2k+n) + g(r-2k-n)], \tag{7}$$

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where

$$g(y) = \frac{2}{\pi} \int_0^\infty \cos yx \frac{\sin \epsilon x}{x} dx = 0, 1, \frac{1}{2} \tag{8}$$

for $| y/\epsilon |$ greater than, less than, or equal to one, respectively [2]. Therefore, the only term which is nonzero in (7) is the term $g(0)$, if it exists.

Then

$$Q_n^r = \frac{1}{2^r} \frac{r!}{\left(\frac{r-n}{2}\right)!\left(\frac{r+n}{2}\right)!}, \qquad r \geqq n \text{ and } r, n \text{ both even or odd,} \tag{9}$$

$$Q_n^r = 0 \quad \text{for } r < n \quad \text{or} \quad r, n \text{ one even, one odd.} \tag{10}$$

We can now write

$$\begin{aligned} r - n &= 2q, \\ r + n &= 2(q + n). \end{aligned} \tag{11}$$

Thus,

$$Q_n^r = \frac{r!}{2^r q!(q+n)!}. \tag{12}$$

Substituting (12) into (5), we obtain

$$A_n(t) = \sum_{q=0}^{\infty} \frac{\left(\frac{t}{2}\right)^{2q+n}}{q!(q+n)!} = I_n. \qquad \text{Q.E.D.} \tag{13}$$

A similar expression can be readily obtained for the coefficients $J_n(t)$. The value of $\epsilon = \frac{1}{2}$ gives the following interesting result. Let us define

$$B_n(t) = \frac{2}{\pi} \int_{n-(1/2)}^{n+(1/2)} \int_0^\infty e^{t \cos x} \cos \nu x \frac{\sin(x/2)}{x} d\nu dx. \tag{14}$$

It can be easily shown that $B_n(t) = I_n(t)$. Consider now the integral

$$\begin{aligned} B(t) &= \frac{2}{\pi} \int_{-\infty}^{\infty} I_\nu \left(\frac{1}{2}, t\right) d\nu \\ &= \frac{2}{\pi} \int_0^\infty d\nu \int_0^\infty 2e^{t \cos x} \cos(\nu x) \frac{\sin(x/2)}{x} dx \\ &= \lim_{y \to 0} \frac{2}{\pi} \int_0^\infty d\nu \int_0^\infty 2e^{t \cos x} \cos(\nu x) \cos(\nu y) \frac{\sin(x/2)}{x} dx \\ &= 2e^{t \cos y} \frac{\sin y/2}{y} \bigg|_{y=0} = e^t \end{aligned}$$

by Fourier's theorem. Clearly $B(t) = \sum_{-\infty}^{\infty} B_n(t)$.

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2. R. S. BURINGTON, *Handbook of Mathematical Tables and Formulas*, 3rd ed., Handbook Publishers Inc., Sandusky, Ohio, 1949. MR **22** #2514.

Footnote to the Evaluation of Certain Complex Elliptic Integrals

By C. D. Sutherland

The formulas for evaluating the elliptic integral of the third kind wi h a complex parameter as given by Byrd and Friedman [1] have been corrected and simplified by Lang and Stevens [2]. There is, however, a further correction necessary in these latter results.

The integral to be evaluated is

$$I = (a_1 + ib_1) \int_0^\phi \frac{d\theta}{(1 - \alpha^2 \sin^2 \theta)\Delta},$$

where α^2 is complex and $\Delta = \sqrt{(1 - k^2 \sin^2 \theta)}$. In the formulas for evaluating I there appears the quantity

$$\tau_2 = \int_0^{p_2} \frac{m_2\, dx}{1 + h_2 x^2} = \frac{m_2}{\sqrt{h_2}} \tan^{-1}(p_2\sqrt{h_2}),$$

where

$$p_2 = \frac{\sin \phi \cos \phi}{(1 + m_2 \sin^2 \phi)\Delta}.$$

We will consider the case where $m_2 \leqq -1$. If this occurs we see that as ϕ goes to $\pi/2$, either $p_2 \to \infty$ $(m_2 = -1)$ and $[\tan^{-1}(p_2\sqrt{h_2})] \to \pi/2$, or $p_2 \to 0$ through negative values $(m_2 < -1)$ and $[\tan^{-1}(p_2\sqrt{h_2})] \to \pi$ (and not to zero). To avoid overlooking this possibility the proper representation for τ_2 is

$$\tau_2 = \frac{-1}{\sqrt{h_2}} \cos^{-1}\left(\frac{\Delta \cos \phi}{\sqrt{(h_2 \sin^2 \phi + \Delta^2 \cos^2 \phi)}}\right) \quad \text{for} \quad m_2 = -1,$$

$$\tau_2 = \frac{m_2}{\sqrt{h_2}} \cos^{-1}\left(\frac{\Delta(1 + m_2 \sin^2 \phi)}{\sqrt{(h_2 \sin^2 \phi \cos^2 \phi + \Delta^2(1 + m_2 \sin^2 \phi)^2)}}\right) \quad \text{for} \quad m_2 \neq -1.$$

It is to be noted, in particular, that the formulas for the real and imaginary parts of the complete integral should contain a term involving τ_2 whenever $m_2 \leqq -1$.

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Equation (416.00′) in reference [2] and equations (416.00) to (419.00) in reference [1] do not contain this term and should be changed.

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1. PAUL F. BYRD & MORRIS D. FRIEDMAN, *Handbook of Elliptic Integrals for Engineers and Physicists*, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, Bd. LXVII, Springer, Berlin, 1954. MR **15**, 702.
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On Davis' Method of Estimating Quadrature Errors

By Y. T. Lo, S. W. Lee, and B. Sun

In quadrature method the error is traditionally estimated in terms of the high derivatives of the integrand. The drawbacks of this method are well known. Some ten years ago, Davis [1], Davis and Rabinowitz [2] introduced an interesting new method in estimating the error for analytic functions in terms of their norms. Briefly, for any function $f(z)$ belonging to $L^2(\mathcal{E}_p)$, where $\mathcal{E}_p$ is a region in the complex z-plane, bounded by an ellipse with foci at $(-1, 0)$ and $(1, 0)$, the error E associated with the quadrature

$$(1) \qquad \int_{-1}^{+1} f(x)\,dx = \sum_{k=0}^{N} a_k f(\lambda_k) + E(f)$$

is bounded by

$$(2) \qquad |E(f)| \leqq \sigma_R \| f \|_{\mathcal{E}_p} .$$

In the above relation,

$$\| f \|_{\mathcal{E}_p} = \left[\iint_{\mathcal{E}_p} | f(z) |^2 \,dx\,dy \right]^{1/2}$$

and σ_R depends only on the ellipse $\mathcal{E}_p$ and the quadrature rule R.

Davis and Rabinowitz [2] have given a short table of σ_R for a few commonly used quadrature rules and various values of the semi-major axis a. To test their results, we have assumed for f a simple trigonometric function whose integral can be easily evaluated and computed. By comparing this with those obtained by various quadratures it turns out that the actual errors are larger than the bound $\sigma_R \| f \|_{\mathcal{E}_p}$. Thus it leads us to believe that their tabulated values of σ_R are in error. In a private communication, Davis agreed with us and encouraged us to recompute their table. Recently we completed this task. The results are tabulated below, where a few more cases and a wider range in semi-major axis a are included. To our disappointment, it is found that these values are much greater than theirs, nearly by a factor of 4.

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