

Gauss Weights and Ordinates for $\int_0^1 f(x)x^2 dx$

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1. Introduction. In the physics of many-particle systems, starting from the Fermi gas model, one frequently encounters integrals with a quadratic weight factor. The most efficient numerical procedure for their evaluation is Gauss quadrature [1]:

$$(1.1) \quad \int_0^1 f(x)x^2 dx = \sum_i^N H_i f(a_i).$$

As is well known, the N -point formula is exact for polynomials $f(x)$ of degree $(2N - 1)$ or less. In the course of some recent work we have evaluated the weights and ordinates for equation (1.1) correct to 15 decimals, up to $N = 20$. This range is adequate for physical applications. Besides, it is only when using a relatively low order formula that there will be a significant gain in accuracy when using these results as opposed to ordinary Gauss quadrature.

2. Procedure. Since the theory of Gaussian quadrature is well known, we state here only the special results needed for this case. The ordinates a_i of equation (1.1) are the zeros of a polynomial, say $P_n(x)$, which is a member of a set of orthogonal polynomials over the interval $(0, 1)$. It is convenient to take

$$(2.1) \quad \int_0^1 P_n(x)P_m(x)x^2 dx = \delta_{nm}/(2n + 3).$$

Then

$$(2.2) \quad P_n(x) = \frac{1}{n!} \frac{\partial^n}{\partial x^n} \{x^{n+2}(x - 1)^n\}.$$

With the particular normalisation chosen here, one finds the recurrence formulae

$$(2.3) \quad P_{n+1}(x) = \frac{2n + 3}{(n + 1)(n + 3)} \left[(2n + 4)x - \frac{(n^2 + 3n + 3)}{(n + 1)} \right] P_n(x) - \frac{n(n + 2)^2}{(n + 3)(n + 1)^2} P_{n-1}(x),$$

$$(2.4) \quad (x^2 - x)P_n'(x) = \frac{n(n + 1)(n + 3)}{2(n + 2)(2n + 3)} P_{n+1}(x) + \frac{n(n + 3)}{2(n + 2)(n + 1)} P_n(x) - \frac{n(n + 2)(n + 3)}{2(2n + 3)(n + 1)} P_{n-1}(x).$$

At first a FORTRAN program was written which found the zeros a_i by Bairstow's method of quadratic factors, having constructed the coefficients of the polynomial $P_n(x)$. It was found advisable, however, to improve the zeros by Newton's method, using the recurrence relations to evaluate the polynomial. The weights H_i were

TABLE 1

a_i			H_i			a_i			H_i		
$N = 1$						$N = 9$					
.75000	00000	00000	.33333	33333	33333	.05875	41357	86726	.00026	12623	46519
$N = 2$						$N = 10$					
.45584	81559	88775	.10078	58820	79825	.12828	99254	25592	.00154	55231	94737
.87748	51773	44559	.23254	74512	53508	.23465	23204	51892	.00646	98890	68559
$N = 3$						$N = 11$					
.29499	77901	11502	.02995	07030	08581	.36039	05113	45290	.01719	75750	46553
.65299	62339	61648	.14624	62692	59866	.49619	28735	85126	.03385	45650	16814
.92700	59759	26850	.15713	63610	64887	.63199	21495	50662	.05288	37887	66964
$N = 4$						$N = 12$					
.20414	85821	03227	.01035	22407	49918	.75771	88038	67476	.06745	22193	81438
.48295	27048	95632	.06863	38871	72923	.86404	97659	49771	.07006	95077	08666
.76139	92624	48137	.14345	87897	99214	.94310	18494	66342	.05627	29364	02808
.95149	94505	53003	.11088	84156	11278	.98903	15475	43826	.02743	40887	10097
$N = 5$						$N = 13$					
.14894	57870	52984	.00411	38252	03099	.04167	48562	13352	.00009	38004	86010
.36566	65273	69113	.03205	56007	22962	.10930	89006	45608	.00096	30205	75543
.61011	36129	34481	.08920	01612	21590	.20121	56414	09085	.00414	00042	63219
.82651	96792	28304	.12619	89618	99911	.31167	57601	37043	.01141	45909	18098
.96542	10600	81785	.08176	47842	85771	.43376	01608	45432	.02358	96635	38807
$N = 6$						$N = 14$					
.11319	43838	22438	.00183	10758	06869	.55980	16001	41138	.03929	02177	98436
.28431	88726	88286	.01572	02971	84945	.68188	20480	93766	.05463	22223	21045
.49096	35868	35248	.05128	95711	29616	.79233	16584	02940	.06418	35593	49756
.69756	30819	77109	.09457	71867	48541	.88421	13733	70605	.06297	95430	00367
.86843	60583	42014	.10737	64997	36781	.95175	03079	05714	.04874	58610	30664
.97409	54449	06333	.06253	87027	26581	.99072	10261	68651	.02330	08500	51388
$N = 7$						$N = 15$					
.08881	68334	36997	.00089	26880	33689	.03579	34647	07328	.00005	95503	59808
.22648	27534	08562	.00816	29256	32305	.09420	49264	55456	.00061	99533	98545
.39997	84867	21007	.02942	22112	89529	.17428	14857	85213	.00272	08421	44104
.58599	78554	02940	.06314	63787	08891	.27175	35470	46927	.00771	63670	23886
.75944	58739	51942	.09173	38032	79795	.38137	61365	41544	.01654	73436	44609
.89691	09708	51951	.09069	88246	12686	.49724	26395	50968	.02890	80518	38727
.97986	72262	26599	.04927	65017	76438	.61310	79900	02310	.04276	47635	70189
$N = 8$						$N = 16$					
.07149	10350	40093	.00046	85177	84035	.72272	65422	00685	.05457	39411	64232
.18422	82964	16716	.00447	45217	13014	.82018	92109	47577	.06025	06074	87605
.33044	77281	75639	.01724	68637	80235	.90024	23744	57386	.05658	41847	49474
.49440	29218	15511	.04081	44263	88544	.95857	21895	45705	.04255	66153	67406
.65834	80085	22798	.06844	71834	21653	.99204	79542	97362	.02003	11125	84749
.80452	48315	11260	.08528	47691	71939	$N = 17$					
.91709	93825	14349	.07681	80932	67223	.03107	11057	71411	.00003	90169	93264
.98390	22404	48079	.03977	89578	06691	.08200	03544	62793	.00041	07208	91791
$N = 9$						$N = 18$					
$N = 10$						$N = 19$					
$N = 11$						$N = 20$					
$N = 12$						$N = 21$					
$N = 13$						$N = 22$					
$N = 14$						$N = 23$					
$N = 15$						$N = 24$					
$N = 16$						$N = 25$					
$N = 17$						$N = 26$					
$N = 18$						$N = 27$					
$N = 19$						$N = 28$					
$N = 20$						$N = 29$					
$N = 21$						$N = 30$					
$N = 22$						$N = 31$					
$N = 23$						$N = 32$					
$N = 24$						$N = 33$					
$N = 25$						$N = 34$					
$N = 26$						$N = 35$					
$N = 27$						$N = 36$					
$N = 28$						$N = 37$					
$N = 29$						$N = 38$					
$N = 30$						$N = 39$					
$N = 31$						$N = 40$					
$N = 32$						$N = 41$					
$N = 33$						$N = 42$					
$N = 34$						$N = 43$					
$N = 35$						$N = 44$					
$N = 36$						$N = 45$					
$N = 37$						$N = 46$					
$N = 38$						$N = 47$					
$N = 39$						$N = 48$					
$N = 40$						$N = 49$					
$N = 41$						$N = 50$					

TABLE 1—Continued

a_i			H_i			a_i			H_i		
$N = 13$ —Cont'd						$N = 17$					
.33729	06864	39611	.01172	82595	93102	.01915	61061	52587	.00000	91808	04200
.44332	53773	70545	.02125	49967	14757	.05090	29834	03455	.00009	93565	25335
.55189	73610	94290	.03294	04516	78433	.09549	45766	85671	.00046	16740	58777
.65793	04873	53925	.04462	46007	55615	.15167	29142	90049	.00141	38837	58956
.75646	71004	12872	.05334	00216	05872	.21782	68969	18779	.00334	65172	91820
.84290	00258	03491	.05611	15371	30827	.29205	49986	14914	.00661	64197	60362
.91318	82198	42415	.05090	98307	12695	.37222	24614	53186	.01141	36422	38637
.96404	66372	99830	.03742	67081	68844	.45602	33219	09524	.01763	87893	31050
.99310	91686	90041	.01740	12286	10062	.54104	69466	46981	.02482	92561	11976
$N = 14$						$N = 18$					
.02722	28792	58817	.00002	62761	61669	.01725	15924	05713	.00000	67100	86125
.07200	42612	53248	.00027	90993	40947	.04589	21044	82683	.00007	29370	08664
.13417	50992	68403	.00126	17838	95157	.08623	01358	89207	.00034	10912	03688
.21122	57615	65439	.00372	48756	75283	.13724	32792	99945	.00105	35502	83957
.30000	90828	30094	.00841	29767	37136	.19761	48501	63583	.00252	07216	38436
.39689	26032	72403	.01569	43322	85073	.26578	26918	64987	.00505	02487	88355
.49791	08602	50022	.02521	61835	36839	.33998	18727	74684	.00885	22782	25095
.59892	86213	02846	.03573	90102	92515	.41829	09473	29082	.01394	38976	53648
.69581	04718	58344	.04525	34322	54461	.49868	18972	49726	.02007	86416	96370
.78459	02429	31104	.05138	99240	39472	.57907	27241	29682	.02672	34243	68960
.86163	34740	40526	.05202	58654	10702	.65738	14021	97992	.03309	59762	02072
.92378	63775	25146	.04591	47721	14671	.73157	98172	13398	.03826	27946	72349
.96850	57401	81342	.03313	95954	69961	.79974	63040	76370	.04128	36658	09958
.99397	11911	28933	.01525	52061	19448	.86011	54291	64831	.04137	71822	87520
$N = 15$						$N = 19$					
.02404	61256	64976	.00001	81287	85570	.01561	71934	48298	.00000	49807	02332
.06371	81782	64481	.00019	39872	63064	.04158	30975	76593	.00005	43426	43132
.11905	04712	84686	.00088	65736	35234	.07823	93182	70860	.00025	55308	58891
.18807	45227	24990	.00265	56629	85774	.12474	68370	16173	.00079	50443	25449
.26830	20269	47477	.00611	10501	10206	.18001	96429	36763	.00191	97884	72692
.35683	53982	31215	.01166	85824	84777	.24276	34808	26702	.00388	97848	12553
.45047	56444	06449	.01929	34165	85275	.31150	82090	85905	.00691	08110	77932
.54583	87642	45523	.02832	65531	74241	.38464	27551	95453	.01106	14406	29037
.63947	83064	12942	.03747	38757	92101	.46045	30372	07912	.01623	18153	21996
.72801	00340	70514	.04499	15558	20616	.53716	22014	47980	.02209	05631	49824
.80823	42937	22346	.04904	12678	94409	.61297	22972	30668	.02809	16079	36103
.87725	17137	84502	.04813	20576	68530	.68610	64325	81782	.03352	50680	70610
.93256	81341	61001	.04153	13076	51604	.75485	04309	04831	.03760	75005	20348
.97218	54380	81740	.02952	76507	56798	.81759	30163	30908	.03959	82482	14881
.99468	09479	97156	.01348	16627	25134	.87286	35901	38440	.03892	23862	22262
$N = 16$						$N = 19$					
.02139	35625	01303	.00001	27784	10570	.91936	67236	33339	.03527	81171	99476
.05677	57142	87754	.00013	75781	03976	.95601	26049	09157	.02870	81497	78704
.10631	40254	51196	.00063	44599	99569	.98194	30216	62099	.01962	08563	40232
.16844	41169	73538	.00192	35319	26067	.99655	61095	96138	.00876	72970	56879
.24117	47420	50859	.00449	48488	51480						
.32217	00825	85439	.00874	75454	95612						
.40882	76146	83497	.01480	37766	57752						
.49836	24438	46812	.02235	71856	49530						
.58789	70358	83140	.03061	20379	17838						
.67455	37899	75870	.03834	80469	51186						
.75554	75682	55489	.04411	26576	75899						
.82827	52405	16998	.04650	62393	97363						
.89039	93907	41229	.04449	65734	72543						
.93992	35622	83362	.03768	58246	81919						
.97525	70323	80565	.02646	08425	17340						
.99527	23128	77065	.01199	94056	24692						

TABLE 1—Continued

a_i			H_i			a_i			H_i		
$N = 20$						$N = 20$					
.01420	42111	59358	.00000	37492	20993	.57185	49590	66743	.02368	22904	14246
.03785	12878	49502	.00004	10391	02087	.64323	91298	97547	.02900	24047	45589
.07130	09850	81249	.00019	38883	09618	.71154	72878	18728	.03355	64314	53475
.11385	86970	85453	.00060	70457	04267	.77532	35806	15695	.03669	74204	72018
.16462	10853	68438	.00147	74444	41971	.83320	87465	18991	.03784	73903	76922
.22250	74079	64513	.00302	24891	70907	.88396	90923	44513	.03658	79165	53227
.28628	43889	84712	.00543	20908	06123	.92652	28082	14715	.03273	46443	17573
.35459	29542	22632	.00881	35560	78243	.95996	30995	38093	.02638	25644	29126
.42597	73418	17173	.01314	09192	56662	.98357	79118	66012	.01791	37523	25739
.49891	61845	17151	.01822	05034	18269	.99686	93162	59256	.00797	57927	36277

then found from the relation [2]

$$(2.5) \quad H_i = -(2n + 4)/(n + 1)(n + 3)P_{n+1}(a_i)P_n'(a_i).$$

The weights and ordinates were checked by applying them to the integrals of x^r , $r = 0, 1, \dots, 2n$.

The initial calculations were carried out on the IBM 7040 at McMaster University. The results presented in Table 1 were computed in double precision on the CDC 1604 at Cornell Computing Center. The latter machine has a 48 bit word length in single precision. The numbers in Table 1 are rounded correct to 15 decimals. The checks mentioned above were satisfied in every case to better than 20 figures, $r \leq (2n - 1)$.

Acknowledgments. We are grateful to the National Research Council of Canada for a research grant, and to the Cornell University Computing Center for permission to use their facilities.

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1. Z. KOPAL, *Numerical Analysis*, 2nd ed., Wiley, New York, 1961; Chapter VII. MR 25 #733.
2. Z. KOPAL, *ibid.*, p. 393.