Mantissa Distributions

By Alan G. Konheim

Let b be an integer, at least 2, and write each positive real number in the form

$$x = mb^c, \tag{1}$$

where m (*the mantissa*) satisfies $1/b \leqq m < 1$ and c (*the characteristic*) is an integer. We define *the product of mantissas*[*] m_1 and m_2 by

$$m_1 * m_2 = \begin{cases} m_1 m_2 & \text{if } 1/b \leqq m_1 m_2 < 1, \\ b m_1 m_2 & \text{if } 1/b^2 \leqq m_1 m_2 < 1/b. \end{cases} \tag{2}$$

Now suppose that M_1 and M_2 are independent, identically distributed random variables, each taking on values in the interval $[1/b, 1)$ such that

$$\Pr(M_1 * M_2 \leqq x) = \Pr(M_1 \leqq x). \tag{3}$$

What are all of the possible choices for the distribution function of M_1 ? The answer is provided by the following

Theorem. $\Pr(M_1 \leqq x) = F_n(x)$ *or* $F_\infty(x)$ $(n = 1, 2, \cdots)$, *where*

$$F_n(x) = \begin{cases} 0 & \text{if } -\infty < x < b^{-1}, \\ 1/n & \text{if } b^{-1} \leqq x < b^{-1+(1/n)}, \\ 2/n & \text{if } b^{-1+(1/n)} \leqq x < b^{-1+(2/n)}, \\ \vdots & \\ 1 & \text{if } b^{-1} \leqq x < \infty, \end{cases} \tag{4}$$

$$= \begin{cases} 0 & \text{if } -\infty < x < b^{-1}, \text{[†]} \\ 1 + 1/n\left[n \dfrac{\log x}{\log b} + 1 \right] & \text{if } b^{-1} \leqq x < 1, \\ 1 & \text{if } 1 \leqq x < \infty, \quad n = 1, 2, \cdots, \end{cases}$$

and

$$F_\infty(x) = \begin{cases} 0 & \text{if } -\infty < x < b^{-1}, \\ 1 + \dfrac{\log x}{\log b} = \displaystyle\int_{1/b}^{x} \frac{du}{u \log b} & \text{if } b^{-1} \leqq x < 1, \\ 1 & \text{if } 1 \leqq x < \infty. \end{cases} \tag{5}$$

Proof. We will write $M_i = b^{-\Theta_i}$ $(i = 1, 2)$, where Θ_1 and Θ_2 are independent, indentically distributed random variables, taking on values in $(0, 1]$. Note that

$$M_1 * M_2 = b^{-(\Theta_1 \dot{+} \Theta_2)},$$

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* If m_i is the mantissa of x_i then $m_1 * m_2$ is the mantissa of $x_1 x_2$.

† [ ] denotes 'the integer part of.'

where $\dot{+}$ denotes addition modulo one. Thus (3) is equivalent to requiring that $\Theta_1 \dot{+} \Theta_2$ and Θ_1 have the same distribution. If we set

$$\phi(n) = E\{e^{2\pi i n \Theta_1}\} = \int_0^1 e^{2\pi i n \theta_1} \, dF_{\Theta_1}(\theta_1),$$

then (3) and the independence of Θ_1, Θ_2 imply

$$\phi(n) = E\{e^{2\pi i n (\Theta_1 \dot{+} \Theta_2)}\} = E\{e^{2\pi i n (\Theta_1 + \Theta_2)}\} = \phi^2(n)$$

so that $\phi(n) = 0$ or 1. Certainly $\phi(0) = 1$. There are two cases to be examined.

Case 1. $\phi(n) = 0$ for all $n \neq 0$.

It follows from the uniqueness theorem for Fourier-Stieltjes series that $dF_{\Theta_1}(d\theta_1) = d\theta_1$ and hence $\Pr(M_1 \leqq x) = F_\infty(x)$.

Case 2. $\phi(n) = 1$ for some $n \neq 0$.

Let m be the smallest positive integer such that $\phi(m) = 1$. Then

$$0 = \int_0^1 (1 - e^{2\pi i m \theta_1}) \, dF_{\Theta_1}(\theta_1) = \int_0^1 (1 - \cos 2\pi m \theta_1) \, dF_{\Theta_1}(\theta_1).$$

It follows that F_{Θ_1} is a 'step function' with points of discontinuity at $\theta_k = k/m$ $(k = 1, 2, \cdots, m)$ and, hence, $\phi(n + m) = \phi(n)$ $(n = 0, \pm 1, \pm 2, \cdots)$. We assert that $\phi(n) = 1$ if and only if $n = km$ for some integer k; for if $\phi(n) = 1$ with $km < n < (k + 1)m$ then $\phi(n - km) = \phi(n) = 1$ while $0 < n - km < m$ contradicting the minimality of m. The uniqueness theorem for Fourier-Stieltjes series now shows that $\Pr(M_1 \leqq x) = F_m(x)$.

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New Primes of the Form $n^4 + 1$

By A. Gloden

This note presents some results of a continuation of the author's systematic factorization of integers of the form $n^4 + 1$ [1].

An electronic computer at l'Institut Blaise Pascal in Paris has been used to find solutions of the congruence $x^4 + 1 \equiv 0 \pmod{p}$ for all primes of the form $8k + 1$ in the interval $10^6 < p < 4 \cdot 10^6$, thereby extending the previous range of such tables listed in [1].

With the aid of these tables, the complete factorization of $n^4 + 1$ has now been effected for all even values of n less than 2040 and for all odd values less than 2397.

Thus, the primality of $\frac{1}{2}(n^4 + 1)$ has been established for the following 116 values of n:

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