$$\int_0^\infty \phi(y)\,dy$$

by some other method. Then, for example,

$$\int_0^\infty \phi(y) \sin y\,dy = \int_0^\infty (1 + \sin y)\phi(y)\,dy - \int_0^\infty \phi(y)\,dy.$$

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18[P, X].—L. S. PONTRYAGIN, V. G. BOLTYANSKII, R. V. GAMKRELIDZE & E. F. MISHCHENKO, *The Mathematical Theory of Optimal Processes*, John Wiley & Sons, Inc., New York, 1962, viii + 360 p., 23 cm. Price $11.95.

One of the major problems of the modern mathematical theory of control processes can be posed in the following terms: "Given a vector differential equation $dx/dt = g(x, y)$, $x(0) = c$, where x represents the state of a physical system at time t, the state vector, and $y(t)$ represents the control vector, determine $y(t)$ so as to minimize a given scalar functional $J = \int_0^T h(x, y, t)\,dt$, where x and y are subject to local constraints of the form $r_i(x, y) \leqq 0, i = 1, 2, \cdots, N$, global constraints of the form $\int_0^T h_i(x, y)\,dt \leqq k_i$, and terminal conditions of the form $f_i(x(T), y(T), T) \leqq 0$." In some cases of importance, T itself depends upon the history of the process, $T = T(x, y)$, and, indeed, may be the quantity we wish to minimize.

The book under review represents a fine and substantial contribution to a new mathematical domain. The major theme of the work is the "maximum principle," an analytic condition which provides important information concerning the structure of extremals, in the terminology of the calculus of variations, or of optimal policies, in the parlance of dynamic programming and control theory.

Since the book is an excellent one that will be widely read and used, it is worthwhile to analyze its objectives and results carefully within the framework of the classical theory of the calculus of variations, and with the desiderata of modern control theory in mind.

In the simplest version of classical variational theory, there are no local or global constraints. The first variation yields the Euler equation, generally a nonlinear differential equation, with two-point boundary conditions. For a variety of reasons, this direct approach is seldom effective computationally. If global constraints are present, Lagrange multipliers may be used to reduce the problem to one without constraints, at the expense of further computational difficulties.

If local constraints of the type indicated above are present, as they are in a large number of the most important classes of processes, the situation is even more complex. This is due to the fact that sometimes the Euler equation holds and sometimes the constraints determine the extremal, or policy. Hence, the analytic and computational difficulties that existed before, as far as effective algorithms for the solution are concerned, are now compounded.

Nevertheless, analogues and extensions of the classical results can be obtained. The pioneering work is that of Valentine [1]. Results of Valentine were used by Hestenes in some unpublished work on constrained trajectories in 1949. In 1961

Berkovitz [2] showed how the maximum principle and results of greater generality could be obtained from Valentine's work combined with the classical calculus of variations.

The principal point of all this discussion is that the maximum principle does not provide us with any analytic approaches which we did not already possess, and does not seem to aid us in the fundamental objective of providing numerical answers to numerical questions. Unfortunately, at the present time, we possess no straightforward approach to the effective analytic solution of constrained variational problems.

This does not diminish the value of the book. Its very elegant presentation of results pertaining to extensions of classical problems and its consideration of processes involving time delays and stochastic elements will have a very stimulating effect upon research in this new field. It will serve the very useful purpose of focussing attention upon new, fascinating, and significant areas of investigation.

Let us now discuss some of the contents of the volume, and present some detailed comments. The authors present a general treatment of the control process formulated above, using the maximum principle (which is, as Berkovitz points out, a restatement of the Weierstrass condition as adapted by Valentine), and discuss some quite interesting examples in detail. In particular, they consider the "bang-bang" control process, where $dx/dt = Ax + y$, y is constrained by the conditions that its components can assume only the values ± 1, and it is desired to reach the origin in minimum time. Following this, they discuss a control process involving retardation (work of Kharatishvili), pursuit processes (work of Kelenozheridze), some interesting applications to approximation theory, problems involving constraints on state variables, and finally some stochastic control processes. The discussion of pursuit and stochastic control seems far more difficult and involved than one based upon the functional equation approach of dynamic programming, and is based upon "open loop" control rather than feedback control.

The authors indicate the intimate relation between dynamic programming and the calculus of variations, and state (p. 7): "... Thus, Bellman's considerations yield a good heuristic method, rather than a mathematical solution of the problem," and again (p. 73): "Thus, even in the simplest examples, the assumptions which must be made in order to derive Bellman's equations do not hold."

These statements provoke some further remarks. In the first place, if one refers to Berkovitz's article, it will be seen that the equations derived from the functional equation approach can be made completely rigorous in a number of cases. In those cases where lines of discontinuity, or more generally, surfaces of discontinuity exist ("switching surfaces"), we have a situation similar to the existence of shocks in hydrodynamics. The classical equations exist on both sides of the shock, and the problem is now that of continuation of the solution from one region to the other.

Perhaps even more important is the following consideration. At the moment, we intend to base computational algorithms on the use of a digital computer. Consequently, there is some merit in formulating control processes in discrete terms from the beginning. If we proceed in this fashion, all problems of existence of extremals vanish, and we face directly the fundamental problems of numerical solution and determination of the structure of optimal policies. Dynamic programming can now be applied in a uniform fashion to the study of deterministic, stochastic, and adaptive control processes. If we so desire, we can establish that various limits exist as the discrete process merges into a continuous one.

The digital computer can be used for mathematical experimentation, with the hope of discerning the structure of optimal policies from the solution of particular problems.

Let us finally note that the authors make no mention of a number of other techniques available for the study of constrained variational problems. Such alternative techniques include: function-space methods [3]; gradient techniques of the type used by Bryson and Kelley [4]; quasilinearization [5]; and techniques based on the Neyman-Pearson lemma [6].

Taking into account all that has been said, there is no question that this book is an important contribution to the theory of control processes: one that must be read by everyone working in that field. Its translation is a fitting tribute to a great mathematician and his distinguished colleagues.

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1. F. A. VALENTINE, "The problem of Lagrange with differential inequalities as added side conditions," *Contributions to the Calculus of Variations 1933–1937*, University of Chicago Press, Chicago, Ill., 1937, p. 407–448.
2. L. D. BERKOVITZ, "Variational methods in problems of control and programming," *J. Math. Anal. Appl.*, v. 3, 1961, p. 145–169.
3. J. P. LASALLE, "Time optimal control systems," *Proc. Nat. Acad. Sci.*, v. 45, 1959, p. 573–577.
4. G. LEITMANN (Editor), *Optimization Techniques*, Academic Press, New York, 1962.
5. R. BELLMAN, H. KAGIWADA & R. KALABA, "A computational procedure for optimal system design and utilization," *Proc. Nat. Acad. Sci.*, v. 48, 1962, p. 1524–1528.
6. R. BELLMAN, I. GLICKSBERG & O. GROSS, *Some Aspects of the Mathematical Theory of Control Processes*, The RAND Corporation, Report 313, 1958.

19[P, X].—YA. Z. TSYPKIN, *Sampling Systems Theory and its Application*, translated from the Russian by R. C. Hutchison and A. Allan, edited by I. Cochrane Pergamon Press, Oxford, 1964 (two volumes), xv + 375 p., 24 cm. Price $30.00.

The avowed intent of this book is to provide methods for analyzing pulse systems and their properties, using insofar as possible techniques that are already familiar in the analysis of continuous systems. Unfortunately, the book is, in my opinion, very unsuccessful in its attempts to meet its aim.

A book of this type should have its subject matter clearly divided into three sections: mathematical material (for example, the discrete Laplace transformation and its application to linear difference equations); systems concepts, such as principles of pulse modulation and digital feedback theory; and, if desired, component descriptions. However, the present book contains a confused mixture of all three. Chapter I, which is supposed to be an introduction to pulse systems, very soon dives into complicated circuit diagrams for the control of electrical machinery, electronic circuits, temperature, and some amazingly intricate mechanical systems, with a very unsatisfactory discussion of modulation theory. Chapter II, which is intended to provide the mathematical background for the sequel, is cluttered with a great number of trivial examples and inelegant theorems; moreover, hardly any