Chebyshev Polynomial Expansions of Complete Elliptic Integrals

By W. J. Cody

1. Introduction. Numerical subroutines for machine computation of the complete elliptic integrals are usually based upon the well-known Gauss arithmetic-geometric mean process [1] or Hastings-type Chebyshev approximations [2], [3], partially because the power-series expansions for these functions converge so slowly. It is well known that the speed of convergence as well as the numerical stability of a power-series expansion can frequently be improved by converting it into a series of Chebyshev polynomials [4]. In this paper we present a number of such converted expansions for the complete elliptic integrals $K(k)$ and $E(k)$.

2. Power-Series Expansions for $K(k)$ and $E(k)$. The complete elliptic integral of the first kind is defined by

$$(1)\qquad K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1 - k^2 \sin^2 \phi)}} = \frac{\pi}{2}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; m\right), \qquad |k| < 1,$$

where $m = k^2$ and ${}_2F_1(a, b; c; z)$ is Gauss' hypergeometric series [5]. Computationally this form is most useful for small m. For $m \approx 1$ there are two other useful expansions: the Legendre form [6]

$$(2)\qquad K(k) = \ln \frac{4}{\sqrt{\eta}}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \eta\right) - K_2(\eta),$$

and a modified Legendre form [3]

$$(3)\qquad K(k) = \frac{1}{2} \ln \frac{1}{\eta}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \eta\right) + \mathcal{K}_2(\eta),$$

where $K_2(\eta)$ and $\mathcal{K}_2(\eta)$ are infinite power series, and $\eta = 1 - m$.

The analogous equations for the complete elliptic integral of the second kind,

$$(4)\qquad \begin{aligned} E(k) &= \int_0^{\pi/2} \sqrt{(1 - k^2 \sin^2 \phi)}\, d\phi, \qquad |k| \leqq 1, \\ &= \frac{\pi}{2}\, {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; m\right), \qquad |k| < 1, \end{aligned}$$

are

$$(5)\qquad E(k) = \ln \frac{4}{\sqrt{\eta}}\, E_1(\eta) + E_2(\eta)$$

for the Legendre form, and

$$(6)\qquad E(k) = \ln \frac{1}{\eta}\, \mathcal{E}_1(\eta) + \mathcal{E}_2(\eta)$$

Received September 16, 1964. Work performed under the auspices of the U. S. Atomic Energy Commission.

for the modified Legendre form, where E_1, E_2, $\mathcal{E}_1$, and $\mathcal{E}_2$ are all infinite power series.

3. Chebyshev Series Expansions. A sufficiently rapidly converging power series can be converted directly into a convergent expansion in shifted Chebyshev polynomials [7]

$$f(x) = \sum_{n=0}^{\infty}{}' a_n T_n^*(x), \qquad 0 \leqq x \leqq 1,$$

where

$$T_n^*(x) = \cos[n \arccos(2x - 1)],$$

and the prime on the summation indicates that only half of the first term is used. All of the expansions of Section 2 are well behaved for values of the variable not greater than $\frac{1}{2}$. Further, the restricted domains $0 \leqq m \leqq \frac{1}{2}$ and $0 \leqq \eta \leqq \frac{1}{2}$ cover the full domain $0 \leqq k^2 \leqq 1$, making it reasonable to convert the expansions into series of shifted Chebyshev polynomials in the variables $2m$ and 2η.

There are two useful expansions corresponding to (1):

$$\text{(7a, b)} \qquad K(k) = \begin{cases} \displaystyle\sum_{n=0}^{\infty}{}' a_n T_n^*(2m), \\ \displaystyle\pi \sum_{n=0}^{\infty}{}' b_n T_n^*(2m), \end{cases} \qquad 0 \leqq m \leqq \frac{1}{2},$$

where (7b) results from conversion of ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; m)$ and the coefficients of (7a) incorporate the multiplication by $\pi/2$. The usefulness of (7b) is pointed up by the expansions corresponding to (2) and (3), where the coefficients b_n occur again:

$$\text{(7c, d)} \quad K(k) = \begin{cases} \displaystyle\ln\frac{16}{\eta}\sum_{n=0}^{\infty}{}' b_n T_n^*(2\eta) - \sum_{n=0}^{\infty}{}' c_n T_n^*(2\eta), \\ \displaystyle\ln\frac{1}{\eta}\sum_{n=0}^{\infty}{}' b_n T_n^*(2\eta) + \sum_{n=0}^{\infty}{}' d_n T_n^*(2\eta), \end{cases} \qquad 0 \leqq \eta \leqq \frac{1}{2}.$$

The expansions corresponding to (4) and (6) are

$$\text{(8a, b)} \quad E(k) = \begin{cases} \displaystyle\sum_{n=0}^{\infty}{}' p_n T_n^*(2m), \qquad 0 \leqq m \leqq \frac{1}{2}, \\ \displaystyle\eta\ln\frac{1}{\eta}\sum_{n=0}^{\infty}{}' q_n T_n^*(2\eta) + \sum_{n=0}^{\infty}{}' r_n T_n^*(2\eta), \qquad 0 \leqq \eta \leqq \frac{1}{2}. \end{cases}$$

The peculiar form of (8b) is necessary to insure proper evaluation of $E(k)$ for $\eta = 0$ ($k^2 = 1$) through the relation

$$\lim_{\eta \to 0} \eta \ln\frac{1}{\eta} = 0.$$

For most floating-point computers, the evaluation of $\ln(1/\eta)$ is accurate for all η except 0. Then the expansion

$$\text{(8c)} \qquad E(k) = \begin{cases} \displaystyle\ln\frac{1}{\eta}\sum_{n=0}^{\infty}{}' s_n T_n^*(2\eta) + \sum_{n=0}^{\infty}{}' r_n T_n^*(2\eta), \qquad 0 < \eta \leqq \frac{1}{2}, \\ 1, \qquad \eta = 0, \end{cases}$$

would be most useful.

TABLE Ia

$$K(k) = \sum{}' a_i T_i^*(2m), 0 \leqq m \leqq \frac{1}{2}$$

i	a_i				
0	3.39777	24271	35543	97735	7142
1	.14003	09050	00041	35907	1349
2	.01333	84495	86727	63989	7455
3	.00157	90046	73690	96955	2093
4	.00020	66230	76906	06232	3139
5	.00002	86422	49149	18138	2405
6		41218	59188	31631	7717
7		6089	55680	48805	5348
8		917	32241	42273	3436
9		140	26862	71140	9898
10		21	70501	60569	0423
11		3	39121	66692	5655
12			53410	81137	5149
13			8469	01565	7412
14			1350	63745	3424
15			216	47417	2743
16			34	84651	1614
17			5	63083	0298
18				91297	2705
19				14847	6531
20				2421	2461
21				395	8102
22				64	8490
23				10	6464
24				1	7511
25					2885
26					476
27					79
28					13
29					2

$\sum{}' a_i = K(1/\sqrt{2}) = 1.85407\ 46773\ 01371\ 91843\ 3850$

$\sum{}'(-)^i a_i = \pi/2$

TABLE Ib

$$K(k) = \pi {\sum}' b_i T_i^*(2m), 0 \leqq m \leqq \frac{1}{2}$$

i	b_i				
0	1.08154	45545	59937	18609	6290
1	.04457	32214	32776	36906	7285
2	.00424	57603	69819	50477	5625
3	.00050	26127	97966	24604	6695
4	.00006	57701	68092	91332	4847
5		91171	11066	72370	1032
6		13120	28529	09857	8893
7		1938	36613	34712	5696
8		291	99279	32665	4288
9		44	64889	07318	4542
10		6	90892	11906	9053
11		1	07945	77920	1563
12			17001	18928	9808
13			2695	77141	0000
14			429	92125	4075
15			68	90586	9287
16			11	09198	9146
17			1	79234	8951
18				29060	8238
19				4726	1548
20				770	7066
21				125	9903
22				20	6421
23				3	3889
24					5574
25					918
26					152
27					25
28					4
29					1

$${\sum}' b_i = K(1/\sqrt{2})/\pi = ({\sum}' c_i + {\sum}' d_i)/ \ln 16$$

$$= 0.59017\ 02995\ 08048\ 11302\ 2669$$

$${\sum}' (-1)^i b_i = 1/2$$

TABLE Ic

$$K(k) = \ln\frac{16}{\eta}\sum{}'b_iT_i^*(2\eta) - \sum{}'c_iT_i^*(2\eta),\ 0 < \eta \leqq \frac{1}{2}$$

i	c_i				
0	.17041	22621	30315	43155	1028
1	.09435	20461	78037	74863	9851
2	.01026	96473	11552	68891	5886
3	.00127	33175	37102	88162	3592
4	.00017	05201	42845	66669	0372
5	.00002	39656	59641	07412	6634
6		34804	63592	88262	8676
7		5175	36666	20927	5519
8		783	38322	70517	9371
9		120	23636	50853	8075
10		18	66072	26752	2056
11		2	92265	83806	8435
12			46124	04947	6578
13			7326	05495	2348
14			1170	05939	6522
15			187	76803	7005
16			30	25882	9099
17			4	89424	3915
18				79422	5883
19				12926	3938
20				2109	3960
21				345	0459
22				56	5639
23				9	2910
24				1	5289
25					2520
26					416
27					69
28					11
29					2

$\sum{}' c_i = 0.19129\ 97184\ 69738\ 22063\ 0544$

$\sum{}' (-)^i c_i = 0$

TABLE Id

$$K(k) = \ln \frac{1}{\eta} \sum{}' b_i T_i^*(2\eta) + \sum{}' d_i T_i^*(2\eta), 0 < \eta \leqq \frac{1}{2}$$

i	d_i				
0	2.82826	59724	42414	17553	4433
1	.02923	11648	80374	51558	6255
2	.00150	21000	07141	47287	6959
3	.00012	02210	38191	71382	0917
4	.00001	18334	83468	55949	4530
5		13123	39681	93150	8514
6		1572	51910	15293	9813
7		198	92541	91411	8240
8		26	19269	85343	1502
9		3	55664	58182	5016
10			49487	43009	3146
11			7023	41195	9203
12			1013	25621	3007
13			148	21045	6753
14			21	93542	3977
15			3	27959	9077
16				49469	4914
17				7520	2573
18				1151	1240
19				177	2896
20				27	4564
21				4	2733
22					6681
23					1049
24					165
25					26
26					4
27					1

$$\sum{}' d_i = 1.44499\ 97981\ 47149\ 89062\ 0972$$

$$\sum{}' (-)^i d_i = \ln 4.$$

TABLE IIa

$$E(k) = \sum{}' p_i T_i^*(2m),\ 0 \leqq m \leqq \frac{1}{2}$$

i	p_i				
0	2.92822	58504	05146	88299	9545
1	− .10983	85572	43451	91176	2083
2	− .00337	07796	33972	36148	2362
3	− .00023	53008	58731	36941	4039
4	− .00002	17641	44792	00668	4306
5	−	23301	64928	43946	8235
6	−	2729	92738	83921	9275
7	−	339	98892	03979	0023
8	−	44	25755	44400	3036
9	−	5	95739	31848	8316
10	−		82323	46149	6100
11	−		11618	87697	1255
12	−		1668	57756	6166
13	−		243	13006	2812
14	−		35	86643	5645
15	−		5	34740	3815
16	−			80463	4863
17	−			12205	7159
18	−			1864	7874
19	−			286	7194
20	−			44	3363
21	−			6	8912
22	−			1	0761
23	−				1687
24	−				266
25	−				42
26	−				7
27	−				1

$\sum{}' p_i = E(1/\sqrt{2}) = 1.35064\ 38810\ 47675\ 50252\ 0175$

$\sum{}' (-)^i p_i = \pi/2$

TABLE IIb

$$E(k) = \eta \ln \frac{1}{\eta} \sum{}' q_i T_i^*(2\eta) + \sum{}' r_i T_i^*(2\eta), 0 \leqq \eta \leqq \frac{1}{2}$$

i	q_i				
0	.56305	88879	96356	77758	1788
1	.03478	63815	91808	15239	9162
2	.00365	57095	20277	50596	6041
3	.00045	19021	79315	92957	8839
4	.00006	05772	21314	44649	5014
5		85266	62411	53353	5347
6		12400	58661	56426	5501
7		1846	18799	15787	0220
8		279	74114	54148	4018
9		42	97300	06565	9237
10		6	67435	62997	9643
11		1	04600	60020	6540
12			16516	63047	0207
13			2624	65105	2908
14			419	36432	4946
15			67	32345	8184
16			10	85276	4985
17			1	75591	0866
18				28502	0895
19				4639	9668
20				757	3400
21				123	9073
22				20	3160
23				3	3376
24					5493
25					906
26					150
27					25
28					4
29					1

$\sum{}' q_i = 2 \sum{}' s_i = 0.32049\ 39989\ 13858\ 43468\ 8701$

$\sum{}' (-)^i q_i = 1/4$

TABLE IIb (*Cont'd.*)

i	r_i				
0	2.23448	68607	29039	00443	7906
1	.11961	91062	84828	73981	8781
2	.00252	62231	70143	06557	7453
3	.00016	38806	47349	18297	3709
4	.00001	47258	39374	78249	4237
5		15529	49525	56853	6576
6		1802	70038	62290	4853
7		223	13086	14184	5666
8		28	91801	65272	7708
9		3	87974	70162	2529
10			53475	80157	8024
11			7531	97267	3412
12			1079	85373	6609
13			157	12789	2994
14			23	15222	5698
15			3	44834	7385
16				51842	8847
17				7858	2084
18				1199	7722
19				184	3607
20				28	4931
21				4	4265
22					6909
23					1083
24					170
25					27
26					4
27					1

$\sum' r_i = 1.23956\ 91251\ 80913\ 92836\ 4238$

$\sum' (-)^i r_i = 1$

TABLE IIc

$$E(k) = \ln \frac{1}{\eta} \sum{}' s_i T_i^*(2\eta) + \sum{}' r_i T_i^*(2\eta), \; 0 < \eta \leqq \frac{1}{2}$$

i	s_i				
0	.14946	13173	97041	23249	5238
1	.07953	59200	87531	32354	3269
2	.00531	87128	51459	88673	8760
3	.00057	75113	87527	97645	2342
4	.00007	26979	10544	54451	3027
5		98588	25599	88472	8901
6		13989	24816	72749	2071
7		2046	58796	80268	6245
8		306	08041	03831	2187
9		46	54518	78784	7767
10		7	17096	49072	8133
11		1	11644	18260	7866
12			17532	31402	4983
13			2773	16211	2621
14			441	33789	5123
15			70	60800	0787
16			11	34811	2377
17			1	83120	0952
18				29654	4041
19				4817	4204
20				784	8193
21				128	1838
22				20	9846
23				3	4426
24					5658
25					932
26					154
27					25
28					4
29					1

$\sum{}' s_i = 0.16024\ 69994\ 56929\ 21734\ 4351$

$\sum{}' (-)^i s_i = 0$

4. Results. The coefficients for the series (7) and (8) were computed on a CDC 3600 in 25-decimal arithmetic. The program for the conversion of the power series to a series of Chebyshev polynomials was a revision of a program originally written by Dr. H. C. Thacher of Argonne National Laboratory. The coefficients, together with theoretical sums, are given in Tables I and II.

All expansions were checked by using the tabulated coefficients to compute complete elliptic integrals for 2000 random arguments in 25-decimal arithmetic. Results were compared against a 25-decimal routine based on the Gauss arithmetic-geometric mean process [1]. In all cases, the maximal errors encountered were within limits imposed by round-off.

Argonne National Laboratory
Argonne, Illinois

1. M. Abramowitz & I. A. Stegun, (Eds.), *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series, Vol. 55, Washington, D. C., 1964.
2. C. Hastings, Jr., *Approximations for Digital Computers*, Princeton Univ. Press, Princeton, N. J., 1955. MR **16**, 963.
3. W. Cody, Jr., "Chebyshev approximations for the complete elliptic integrals K and E," *Math. Comp.*, v. 19, 1965, pp. 105–112.
4. C. W. Clenshaw, *Chebyshev Series for Mathematical Functions*, National Physical Laboratory Mathematical Tables, Vol. 5, Department of Scientific and Industrial Research, Her Majesty's Stationery Office, London, 1962. MR **26** #362.
5. A. Erdélyi et al., *Higher Transcendental Functions*, Vol. II, McGraw-Hill, New York, 1953. MR **15**, 419.
6. J. R. Airey, "Toroidal functions and the complete elliptic integrals," *Philos. Mag.*, v. 19, 1935, pp. 177–188.
7. H. C. Thacher, Jr., "Conversion of a power series to a series of Chebyshev polynomials," *Comm. ACM*, v. 7, 1964, pp. 181–182.