Abscissas, Coefficients, and Error Term for the Generalized Gauss-Laguerre Quadrature Formula Using the Zero Ordinate

By C. R. Cassity

1. Introduction. Several tables of abscissas and coefficients for the generalized Gauss-Laguerre quadrature formula not using the zero ordinate have been issued recently [1–5]. Such formulas are all of odd degree of precision, and the error term is known [2]. Formulas using the zero ordinate are of even degree of precision; abscissas and coefficients have been tabulated only very briefly by Burnett [6] and copied by Kopal [7], and more elaborately but only for $s = 0$ by Krylov and Fedenko [8]; no formula for the error term has been published—it is derived in Section 3 below.

Table I in Section 2 gives abscissas and coefficients for $s = 0, -1/3, -1/2, -2/3$, with $n = 2(1)16$, thus complementing three of the tables of [2] and [3] by providing the abscissas and coefficients for the quadratures of even degree of precision. A less accurate table covering

$$s = -.99(.01) -.90, -.80, -.75, -.70(.1) -.3, -.25, -.2, -.1, 0(1)10$$

may be obtained by requesting reference [9]. All the tables include values of $H_j e^{a_j}$ for convenience in the integration of functions not explicitly containing the factor e^{-x}.

In selecting a formula for practical use in numerical integration one is usually concerned with two matters: (1) a minimum amount of calculational labor and (2) an error which is acceptably small. The amount of calculational labor often is controlled by the number of times that $f(x)$ must be evaluated, and the only significant difference between one evaluation and another is that $f(0)$ may be zero or more easily calculated than other values. If the labor of calculating $f(0)$ may be neglected, the number of evaluations of $f(x)$ for the formula using the zero ordinate and of degree of precision $2n - 2$ is $n - 1$, the same as the number of evaluations for the formula not using the zero ordinate and of lower degree of precision $2n - 3$; otherwise the number of evaluations is n and equals that not using the zero ordinate of degree of precision $2n - 1$. Anticipating the result of Section 3, we see that the error terms of the degrees of precision $2n - 3$, $2n - 2$ and $2n - 1$ are, respectively,

$$\frac{(n-1)!\,\Gamma(n+s)f^{(2n-2)}(\eta)}{(2n-2)!}, \qquad \frac{(n-1)!\,\Gamma(n+s+1)f^{(2n-1)}(\eta)}{(2n-1)!},$$

$$\frac{n!\,\Gamma(n+s+1)f^{(2n)}(\eta)}{(2n)!}.$$

Disregarding the unknown behavior of the higher derivatives, the ratio of the second coefficient to the first is $(n + s)/(2n - 1)$; the ratio of the third coefficient

Received May 11, 1964. Revised August 3, 1964.

to the second is 1/2. Thus, for the values of n and s included in Table I, the first ratio is nearly 1/2; the error terms decrease in a uniform way; if the labor to evaluate $f(0)$ is negligible, one should select the formula with the zero ordinate, otherwise use the formula with the same number of ordinates not including $x = 0$. For the larger values of s, such as some of those in reference [9], the first ratio becomes larger than 1, and the selection of a most suitable formula cannot be stated so simply.

2. Numerical Values of Abscissas and Coefficients. The abscissas a_j and the coefficients H_j given in Table I following are derived from the requirement of maximum degree of precision ($E = 0$ for $f(x)$ a polynomial of as high degree as possible) of the quadrature formula

$$\int_0^\infty x^s e^{-x} f(x)\, dx = H_1 f(0) + \sum_{j=2}^{n} H_j f(a_j) + E. \tag{1}$$

Since the a_j and H_j, including H_1 but excluding $a_1 = 0$, constitute a set of $2n - 1$ parameters, the degree of precision is $2n - 2$.

Numerical values have been calculated by using the formulas given by Kopal [7]. Equivalent formulas may be found in Mineur [10] and in Krylov and Fedenko [8]. (Facile comparison of formulas and numerical values is impeded by differences of notation. E. g., Kopal's H_1 does not include the factor $1 + s$ which is included by Krylov and Fedenko and in formula (1) herein; Krylov and Fedenko begin formula (1) with $Af(0)$ and run the summation from 1 to n so that their n is one smaller than the present n which is the same as Kopal's.) Krylov and Fedenko [8] have observed that the $a_j \neq 0$ of (1) are the same as the a_j of the formula not using the zero ordinate, but for a value of s increased by unity:

$$\int_0^\infty x^{s+1} e^{-x} f(x)\, dx = \sum_{j=2}^{n} K_j f(a_j)$$

and that $K_j = H_j a_j$ ($j = 2, \cdots, n$).

The abscissas of (1) are roots of the polynomial

$$\Lambda_n^s(x) \equiv L_n^s(x) - (n + s) L_{n-1}^s(x) = 0, \tag{2}$$

where $L_n^s(x)$ represents the generalized Laguerre (or Sonine) polynomial. As given in [7],

$$H_j = \frac{\Gamma(n)\Gamma(n+s)}{(n+s)\{L_{n-1}^s(a_j)\}^2},$$

but the present notation incorporates the factor $1 + s$ into H_1.

For each combination of n and s, the accuracy of calculated a's and H's has been checked by the relations

$$\sum_{j=1}^{n} a_j = (n-1)\cdot(n+s), \qquad \prod_{j=2}^{n} a_j = \frac{\Gamma(n+s+1)}{\Gamma(s+2)} \quad \text{and} \quad \sum_{j=1}^{n} H_j = \Gamma(s+1).$$

In no case do the calculated and theoretical values differ in more than the last two digits. Since there are obvious cases (e.g., H_1 at $s = 0$, $n = 11$) in which the calculated values are incorrect in the last two digits, it is expected that only 14 digits of H are reliable. The a's are probably good to 15 digits.

TABLE I

S= 0.0

N	A	H	H X EXP(A)
14	0.000000000000000D 00	7.142857142857178D-02	7.142857142857178D-02
	2.625883981710845D-01	3.397031747598451D-01	4.417131188909584D-01
	8.835503073877194D-01	3.312612154569891D-01	8.014780151955862D-01
	1.869033815199808D 00	1.807621379161582D-01	1.171705697256321D 00
	3.232418699451171D 00	6.149540307525772D-02	1.558347314250822D 00
	4.993575607074209D 00	1.334983673277505D-02	1.968603647614237D 00
	7.180610494100796D 00	1.835971727099759D-03	2.411934464800880D 00
	9.832808251019998D 00	1.557032689822949D-04	2.901563608590766D 00
	1.300562240025862D 01	7.770654259463208D-06	3.457224869491441D 00
	1.677961364117566D 01	2.121403111337379D-07	4.110709761449397D 00
	2.127791175544757D 01	2.824594525851680D-09	4.918535099305593D 00
	2.670503409892745D 01	1.513671259723815D-11	5.996260793219731D 00
	3.345278496576768D 01	2.263302388073614D-14	7.640160755348884D 00
	4.252444756601822D 01	3.762192860177174D-18	1.105538773563426D 01
15	0.000000000000000D 00	6.666666666666698D-02	6.666666666666698D-02
	2.450330150930988D-01	3.225409545444101D-01	4.120988031631485D-01
	8.240822700478656D-01	3.776690703167475D-01	7.470158380795286D-01
	1.741872400764467D 00	1.910197222697775D-01	1.090344610237242D 00
	3.009132459017929D 00	7.137636117082454D-02	1.446785094316890D 00
	4.641634066590660D 00	1.756613222046359D-02	1.821848290753420D 00
	6.661341496627173D 00	2.843563441151851D-03	2.222525620444962D 00
	9.098302703770528D 00	2.973320339597119D-04	2.658179844574904D 00
	1.199370343318572D 01	1.942738183535109D-05	3.142052915280345D 00
	1.540498803105152D 01	7.537052122997047D-07	3.694050381958025D 00
	1.941499282816571D 01	1.608066381609186D-08	4.346385309010819D 00
	2.414975795202567D 01	1.675895585089107D-10	5.156482815292279D 00
	2.981810511967375D 01	7.006205186519444D-13	6.241965534926862D 00
	3.681962425700088D 01	8.078467831675617D-16	7.904759124043877D 00
	4.617743001698500D 01	1.003040250495504D-19	1.137421428241558D 01
16	0.000000000000000D 00	6.250000000000028D-02	6.250000000000028D-02
	2.296805054251354D-01	3.069586969659922D-01	3.862148058002698D-01
	7.721449103754117D-01	3.232021679961100D-01	6.995399781291305D-01
	1.631053099067446D 00	1.995846215419095D-01	1.019728213677492D 00
	2.815144590012255D 00	8.089356722467788D-02	1.350565801388402D 00
	4.337164077337561D 00	2.217763872150845D-02	1.696374479866483D 00
	6.214642764559239D 00	4.124820342107398D-03	2.062481668086591D 00
	8.471163981346716D 00	5.142914494969424D-04	2.455777840838193D 00
	1.113833196575080D 01	4.196820236533496D-05	2.885602314795530D 00
	1.425891002162424D 01	2.159999485673012D-06	3.365267474777699D 00
	1.789205343816948D 01	6.642001556226309D-08	3.914885378424358D 00
	2.212262017483362D 01	1.126962155133980D-09	4.567107133921378D 00
	2.707931149904755D 01	9.341303337949287D-12	5.380257167863485D 00
	3.297497355239741D 01	3.092580705417929D-14	6.473961266592954D 00
	4.021658371149655D 01	2.790069220343955D-17	8.155589629522774D 00
	4.984622170855657D 01	2.626753027126382D-21	1.167772579010531D 01

S=-0.5

N	A	H	H X EXP(A)
14	0.000000000000000D 00	4.235772880150611D-01	4.235772880150611D-01
	1.796131365633884D-01	7.090346081995104D-01	8.485402113026148D-01
	7.208202589466973D-01	4.147409615020201D-01	8.527567997189405D-01
	1.630899655950554D 00	1.683461621681045D-01	8.599910692095132D-01
	2.922597921220351D 00	4.683186725068391D-02	8.705818807929681D-01
	4.615178476472022D 00	8.762559671585114D-03	8.850698236636876D-01
	6.736185972285235D 00	1.073539231376227D-03	9.042870609202162D-01
	9.324365594647539D 00	8.293516008768314D-05	9.295210136661592D-01
	1.243461777443965D 01	3.830561801190605D-06	9.628272152437620D-01
	1.614688493695545D 01	9.790800819399118D-08	1.007676762333293D 00
	2.058352207215326D 01	1.231006315646364D-09	1.070459732667825D 00
	2.594791911981655D 01	6.268235227246092D-12	1.164617673561774D 00
	3.262986350068336D 01	8.941544914786263D-15	1.325587611760693D 00
	4.162753157986593D 01	1.419155859410274D-18	1.700737763785067D 00
15	0.000000000000000D 00	4.089711746352321D-01	4.089711746352321D-01
	1.674156996761943D-01	6.928378781433420D-01	8.191045483952511D-01
	6.715780886047635D-01	4.202853888984228D-01	8.226345587353285D-01
	1.518355667079083D 00	1.815376844311591D-01	8.286674485551988D-01
	2.717957512210810D 00	5.527931118404562D-02	8.374455369702599D-01
	4.285649687649694D 00	1.169100709054386D-02	8.493485205225975D-01
	6.242930455366954D 00	1.681585012280395D-03	8.649461612752070D-01
	8.619406078996830D 00	1.598184070904479D-04	8.850897170024954D-01
	1.145582273111915D 01	9.646146995295216D-06	9.110757212738669D-01
	1.480915730405410D 01	3.498458259899963D-07	9.449568890067748D-01
	1.876170420982646D 01	7.040624655201392D-09	9.901837654086081D-01
	2.343880200179692D 01	6.968260322616879D-11	1.053093930652507D 00
	2.904822495602604D 01	2.780214542899216D-13	1.146996567476616D 00
	3.598765653181121D 01	3.069000889210654D-16	1.306885870739692D 00
	4.527533907578178D 01	3.648921705534494D-20	1.678782648400224D 00
16	0.000000000000000D 00	3.957785560986114D-01	3.957785560986114D-01
	1.567706505644788D-01	6.775453157882182D-01	7.925432574966491D-01
	6.286539005223592D-01	4.242655773375337D-01	7.955339475246168D-01
	1.420451593996079D 00	1.934358963217410D-01	8.006290782091156D-01
	2.540472136213071D 00	6.369446146258428D-02	8.080062129603387D-01
	4.001034262589769D 00	1.496559982809541D-02	8.179395917469434D-01
	5.819276712175230D 00	2.467366614601913D-03	8.308324468120396D-01
	8.018414190110819D 00	2.790418079031849D-04	8.472709445770997D-01
	1.062969127755461D 01	2.099720551517253D-05	8.681160090725875D-01
	1.369550173656081D 01	1.008742511793916D-06	8.946664568468321D-01
	1.727459490821158D 01	2.922397668974343D-08	9.289675815176628D-01
	2.145134346051718D 01	4.704984006420744D-10	9.744470513669433D-01
	2.635380354934771D 01	3.720747042903946D-12	1.037388917092528D 00
	3.219383847501827D 01	1.179911299750780D-14	1.130982177748678D 00
	3.937661275322736D 01	1.022121801585435D-17	1.289876963434418D 00
	4.893954039339065D 01	9.237936788928421D-22	1.658621634862379D 00

TABLE I (continued)

S=-1/3

N	A	H	H X EXP(A)
2	0.0000000000000000D 00	8.1247076365584020D-01	8.1247076365584020D-01
	1.6666666666666667D 00	5.4164717577056010D-01	2.8677455829824210D 00
3	0.0000000000000000D 00	6.0935307274188040D-01	6.0935307274188040D-01
	1.0336735048112150D 00	7.0407223365662630D-01	1.9794107142620170D 00
	4.2996598285221180D 00	4.0692633027894090D-02	2.9980186453358670D 00
4	0.0000000000000000D 00	4.9856160497062960D-01	4.9856160497062960D-01
	7.5427531616805340D-01	7.4344041431847900D-01	1.5806065372885900D 00
	2.9691486965685580D 00	1.1000382383605530D-01	2.1423611135773180D 00
	7.2765759872633900D 00	2.1120963012376950D-03	3.0541510651619880D 00
5	0.0000000000000000D 00	4.2733851854625410D-01	4.2733851854625410D-01
	5.9488311662037330D-01	7.3955909842505410D-01	1.3406868179067550D 00
	2.2945373791794590D 00	1.7645255959386210D-01	1.7503821863144030D 00
	5.3360190193713370D 00	1.0677641072000380D-02	2.2175781481067290D 00
	1.0441227151495500D 01	9.0121789231199600D-05	3.0860086680678540D 00
6	0.0000000000000000D 00	3.7706339871728320D-01	3.7706339871728320D-01
	4.9145763660772590D-01	7.1970508473236060D-01	1.1764999435293760D 00
	1.8767642380823950D 00	2.3101096616518670D-01	1.5090406851292110D 00
	4.2734123312278030D 00	2.5549454515705550D-02	1.8335849121934280D 00
	7.9650375423223410D 00	7.8563984475303080D-04	2.2614935526028460D 00
	1.3726661585093070D 01	3.3954511131263730D-06	3.1067825941572640D 00
7	0.0000000000000000D 00	3.3935705884555520D-01	3.3935705884555520D-01
	4.1881027022623220D-01	6.9430090779546360D-01	1.0554428506486170D 00
	1.5902573254038720D 00	2.7327541095582960D-01	1.3404188830680840D 00
	3.5813953423344940D 00	4.4413906440010040D-02	1.5955088153585600D 00
	6.5394117960968900D 00	2.7225229173370800D-03	1.8836577732933710D 00
	1.0772776604109710D 01	4.8015230576565990D-05	2.2905324704886190D 00
	1.7097348661828800D 01	1.1724163008974330D-07	3.1215193189576320D 00
8	0.0000000000000000D 00	3.0984774938072430D-01	3.0984774938072430D-01
	3.6494021273143360D-01	6.6763780047487850D-01	9.6168410576927400D-01
	1.3807727409175680D 00	3.0514832800073960D-01	1.2138722320142840D 00
	3.0892520789697140D 00	6.5015669203768030D-02	1.4277861609998090D 00
	5.5782588061849130D 00	6.2315979180497970D-03	1.6489460310685340D 00
	9.0101736825321690D 00	2.3422559593996870D-04	1.9173573556703810D 00
	1.3711333314848400D 01	2.5650591855715210D-06	2.3112865314863490D 00
	2.0531935830482470D 01	3.7931152489998060D-09	3.1325821748508940D 00
9	0.0000000000000000D 00	2.8601330712066850D-01	2.8601330712066850D-01
	3.2338288777975770D-01	6.4153650183759830D-01	8.8647149546077500D-01
	1.2206197099544380D 00	3.2879568728296800D-01	1.1143830991922080D 00
	2.7193024077660770D 00	8.5754263703052380D-02	1.3008695637881470D 00
	4.8766702434788880D 00	1.1304047544550630D-02	1.4830129803225420D 00
	7.7910411511600510D 00	6.9682960754672920D-04	1.6855155957981740D 00
	1.1635417945708010D 01	1.7178569739259580D-05	1.9417119828845510D 00
	1.6750412507491190D 01	1.2364363264345090D-07	2.3269308515877000D 00
	2.4016486479994910D 01	1.1664669229230290D-10	3.1412317695729580D 00

S=-2/3

N	A	H	H X EXP(A)
2	0.0000000000000000D 00	2.0092039010308120D 00	2.0092039010308120D 00
	1.3333333333333333D 00	6.6973463367693660D-01	2.5407507777375930D 00
3	0.0000000000000000D 00	1.7221747723121260D 00	1.7221747723121260D 00
	8.0580810168138650D-01	9.1682611753468180D-01	2.0523195800890150D 00
	3.8608585649852800D 00	3.9937644860942910D-02	1.8972825874904310D 00
4	0.0000000000000000D 00	1.5499572950809140D 00	1.5499572950809140D 00
	5.8046512131708260D-01	1.0168270098237180D 00	1.8169370166024140D 00
	2.6321997616344630D 00	1.1034091679285670D-01	1.5342156899106880D 00
	6.7873351170484540D 00	1.8133130102621610D-03	1.6075887774387610D 00
5	0.0000000000000000D 00	1.4307298108439200D 00	1.4307298108439200D 00
	4.5424410475084620D-01	1.0575771306007290D 00	1.6656653792296020D 00
	2.0192169205841790D 00	1.8127691565851480D-01	1.3654546150368220D 00
	4.9379857204000750D 00	9.2843200985859010D-03	1.2950604942362910D 00
	9.9218865875982300D 00	7.0357506001027960D-05	1.4332799692942670D 00
6	0.0000000000000000D 00	1.3413091976661760D 00	1.3413091976661760D 00
	3.7330742627062240D-01	1.0712722537007210D 00	1.5560559622529360D 00
	1.6435057353058020D 00	2.4320514206800050D-01	1.2581668108959050D 00
	3.9353349642709180D 00	2.2530576005442800D-02	1.1530988904472270D 00
	7.5274254517543930D 00	6.1890335388525410D-04	1.1501173971321590D 00
	1.3187093089064930D 01	2.4619135294870330D-06	1.3132716101447570D 00
7	0.0000000000000000D 00	1.2707139767363790D 00	1.2707139767363790D 00
	3.1692792573424330D-01	1.0715508566727350D 00	1.4711360472890100D 00
	1.3877180968233320D 00	2.9470393402123890D-01	1.1804952534469130D 00
	3.2867731719785390D 00	3.9769616382428200D-02	1.0640912050377620D 00
	6.1582619626772470D 00	2.1650029204313690D-03	1.0231932702261760D 00
	1.0307180703084080D 01	3.5067962077419810D-05	1.0501763447452460D 00
	1.6543138139702560D 01	8.0012465250523810D-08	1.2239150045153540D 00
8	0.0000000000000000D 00	1.2129542505210890D 00	1.2129542505210890D 00
	2.7537861189435200D-01	1.0648546890182930D 00	1.4024447441250860D 00
	1.2017146398701870D 00	3.3679997686495590D-01	1.1201342840189280D 00
	2.8278507250826190D 00	5.9148089528762370D-02	1.0001397470140050D 00
	5.2396485605770250D 00	5.0075118645028490D-03	9.4443580596834380D-01
	8.5973972869012540D 00	1.7225278885792340D-04	9.3318653242700970D-01
	1.3224770231774970D 01	1.7616621262455860D-06	9.7581469440750100D-01
	1.9966573277232920D 01	2.4591666903089610D-09	1.1538797836746490D 00
9	0.0000000000000000D 00	1.1644360805002450D 00	1.1644360805002450D 00
	2.4347866507469960D-01	1.0544753153425470D 00	1.3451721223366180D 00
	1.0601184237456460D 00	3.7104098739241540D-01	1.0719861786318410D 00
	2.4842245865329380D 00	7.9272942579614900D-02	9.5062670224155870D-01
	4.5716056142426790D 00	9.1857851517069850D-03	8.8825850619351020D-01
	7.4191594335641770D 00	5.1638043784168920D-04	8.6112436029617870D-01
	1.1198303590638250D 01	1.1862604655033500D-05	8.6604710244324930D-01
	1.6247476071517110D 01	8.0626396646322730D-08	9.1762757011786690D-01
	2.3442309281351160D 01	7.2331475090285250D-11	1.0969652900165180D 00

TABLE I (continued)

S=-1/3

N	A	H	H X EXP(A)
10	0.000000000000000D 00	2.662882514571746D-01	2.662882514571746D-01
	2.903415031448485D-01	6.167476799482740D-01	8.245200807852822D-01
	1.094065493513544D 00	3.460873199299447D-01	1.033551911269156D 00
	2.430203011216914D 00	1.056500896958163D-01	1.200310562371438D 00
	4.338342542825345D 00	1.771901240597386D-02	1.356930921389556D 00
	6.883366250836575D 00	1.558940017546180D-03	1.521378887153038D 00
	1.017083737564347D 01	6.552800037803314D-05	1.712243254716826D 00
	1.438197387563279D 01	1.112474435087059D-06	1.960206985994255D 00
	1.986951990584242D 01	5.493417763053866D-09	2.339189902878453D 00
	2.754135004134409D 01	3.443551949839671D-12	3.148204869368376D 00
11	0.000000000000000D 00	2.496452357411009D-01	2.496452357411009D-01
	2.634373734171363D-01	5.935368480362505D-01	7.724263554924564D-01
	9.914687842241054D-01	3.585066975973954D-01	9.662437449696419D-01
	2.197646424239986D 00	1.241725657894196D-01	1.118024627867434D 00
	3.910590813124292D 00	2.516808764343854D-02	1.256603395088469D 00
	6.175721286308818D 00	2.903539442357028D-03	1.396398605323042D 00
	9.063245460671018D 00	1.795290939841162D-04	1.549716763454038D 00
	1.268423763958666D 01	5.370682417509392D-06	1.732702037908083D 00
	1.722669661203245D 01	6.517157012394923D-08	1.974774289216936D 00
	2.305408302798972D 01	2.283691792564076D-10	2.349083938554114D 00
	3.109953924507246D 01	9.828748464377321D-14	3.153962349982771D 00
12	0.000000000000000D 00	2.353797936987526D-01	2.353797936987526D-01
	2.411034757169637D-01	5.719453041410767D-01	7.278877453749104D-01
	9.065758063241820D-01	3.671945000360707D-01	9.091112617416925D-01
	2.006304258256358D 00	1.410824502641204D-01	1.049058874660926D 00
	3.561699865469257D 00	3.333499213693067D-02	1.174159117450040D 00
	5.605963144637787D 00	4.766526645848935D-03	1.296704101796030D 00
	8.188612653324524D 00	3.960950891885130D-04	1.425833281841988D 00
	1.138388207136118D 01	1.788044190344232D-05	1.571576094719298D 00
	1.530730727838377D 01	3.934488730323291D-07	1.748909712051295D 00
	2.015273507815939D 01	3.514658389616639D-09	1.986574056167259D 00
	2.629333398120385D 01	8.981668265033697D-12	2.357256648840060D 00
	3.468581572049605D 01	2.726872742492044D-15	3.158808067477559D 00
13	0.000000000000000D 00	2.229913835040819D-01	2.229913835040819D-01
	2.222650558428453D-01	5.519122199459242D-01	6.892844755850908D-01
	8.351456828118352D-01	3.730184808237525D-01	8.598634930805251D-01
	1.845989513920344D 00	1.563178385986735D-01	9.901741882165405D-01
	3.271257491433280D 00	4.193597780442756D-02	1.104780121121994D 00
	5.136038900119339D 00	7.142799072139205D-03	1.214567865868441D 00
	7.476790093323536D 00	7.511059468999612D-04	1.326874599458941D 00
	1.034612947951228D 01	4.652782415760887D-05	1.448704087665436D 00
	1.382151414343354D 01	1.579483256703054D-06	1.588994341168847D 00
	1.802225679880654D 01	2.624597574322661D-08	1.762095338428520D 00
	2.314740321388801D 01	1.767854206265291D-10	1.996346544370289D 00
	2.957906895366981D 01	3.370123781454423D-13	2.364135033869865D 00
	3.829614067323862D 01	7.383969293640281D-17	3.162950931840609D 00

S=-2/3

N	A	H	H X EXP(A)
10	0.000000000000000D 00	1.122849077625237D 00	1.122849077625237D 00
	2.182119107972991D-01	1.042192973851369D 00	1.296332381969025D 00
	9.486105257311501D-01	3.988924344280386D-01	1.029987884307257D 00
	2.216542456647600D 00	9.922586195691816D-02	9.104519660221577D-01
	4.060543109564098D 00	1.456706185568937D-02	8.449741644807109D-01
	6.544449106379705D 00	1.164876174517427D-03	8.100240185684751D-01
	9.772769203066945D 00	4.551664870735666D-05	7.987852121354824D-01
	1.392547249702311D 01	7.287401944402857D-07	8.134457370160917D-01
	1.935339482101571D 01	3.425031820514021D-09	8.704366875080625D-01
	2.696003636980437D 01	2.052961602549626D-12	1.049452593440790D 00
11	0.000000000000000D 00	1.086628139637327D 00	1.086628139637327D 00
	1.977019834819415D-01	1.029019102918209D 00	1.253950864559731D 00
	8.584657484810526D-01	4.215944358348636D-01	9.947680037243140D-01
	2.001773852088653D 00	1.184506028273070D-01	8.767920700676204D-01
	3.655450539550742D 00	2.093738546294211D-02	8.099616329413598D-01
	5.864116696875913D 00	2.188785217197201D-03	7.708265583098581D-01
	8.697207194987259D 00	1.255070738146169D-04	7.513067607312678D-01
	1.226496603801721D 01	3.534804369019133D-06	7.498466475557149D-01
	1.675433757126021D 01	4.079480891621654D-08	7.707637097600105D-01
	2.252707531295643D 01	1.368617512477805D-10	8.311254759326562D-01
	3.051223839563390D 01	5.656861634154009D-14	1.008956219613348D 00
12	0.000000000000000D 00	1.054668488471524D 00	1.054668488471524D 00
	1.807198939364439D-01	1.015506417511243D 00	1.216657464548472D 00
	7.840513969941538D-01	4.401449560578798D-01	9.649619106721117D-01
	1.825452799349170D 00	1.366387716639333D-01	8.479261457897452D-01
	3.325720662060527D 00	2.806284399959596D-02	7.806813670205612D-01
	5.317427425586192D 00	3.626156102744017D-03	7.392237047607304D-01
	7.849496953671552D 00	2.788225282337141D-04	7.150247479359975D-01
	1.099555936262615D 01	1.182899524885066D-05	7.051128250605826D-01
	1.487046268941288D 01	2.472582638265177D-07	7.100855987240116D-01
	1.966713044679923D 01	2.113894474211108D-09	7.352068273958345D-01
	2.575717156994362D 01	5.195618256130349D-12	7.976910445854071D-01
	3.409347346628675D 01	1.520167554611850D-15	9.738652162938362D-01
13	0.000000000000000D 00	1.026163934729052D 00	1.026163934729052D 00
	1.664267854892369D-01	1.002021867575857D 00	1.183465042139289D 00
	7.215631711321526D-01	4.553375343021190D-01	9.369239809504899D-01
	1.677998150780533D 00	1.536427079255892D-01	8.227299180805240D-01
	3.051710904304759D 00	3.572306985935773D-02	7.555965720803695D-01
	4.867285671420100D 00	5.484880257392524D-03	7.128570675057861D-01
	7.160705853026939D 00	5.325749227159167D-04	6.858604364823525D-01
	9.984126177012802D 00	3.095236674670986D-05	6.710343743804180D-01
	1.341450962591702D 01	9.968266063752669D-07	6.675243756388865D-01
	1.757057714252586D 01	1.583953791101283D-08	6.769353667308514D-01
	2.265054499870215D 01	1.026042831119030D-10	7.049725312950368D-01
	2.903508292357650D 01	1.888102449683636D-13	7.687796008043340D-01
	3.769946859611194D 01	3.998175984153325D-17	9.430464577740386D-01

Table I (continued)

S=-1/3

N	A	H	H X EXP(A)
14	0.0000000000000000D 00	2.121137550404678D-01	2.121137550404678D-01
	2.061602029803074D-01	5.333335750269010D-01	6.554403341970432D-01
	7.741975717439058D-01	3.766377694147656D-01	8.168712334738746D-01
	1.709648434670987D 00	1.699189388640276D-01	9.391450384184235D-01
	3.025454215625146D 00	5.073518779115902D-02	1.045315387162455D 00
	4.741082761933424D 00	9.997257387547519D-03	1.145267549142494D 00
	6.884269193641857D 00	1.274804526471951D-03	1.245213210845401D 00
	9.493941726866768D 00	1.016998380321018D-04	1.350475671097374D 00
	1.262518789808916D 01	4.823805284282288D-06	1.467030712875353D 00
	1.635815866947270D 01	1.261064831147308D-07	1.603228624998974D 00
	2.081546961277531D 01	1.616880266308357D-09	1.773051913810439D 00
	2.620088895767505D 01	8.377673674071175D-12	2.004587117612769D 00
	3.290487976542947D 01	1.214348104812313D-14	2.370013987592845D 00
	4.192732765576254D 01	1.957843071808683D-18	3.166539577080584D 00
15	0.0000000000000000D 00	2.024722207204467D-01	2.024722207204467D-01
	1.922336396574577D-01	5.160910131850764D-01	6.254783842320220D-01
	7.215733679943919D-01	3.785544508014050D-01	7.789394305112867D-01
	1.592232369214383D 00	1.819806985834731D-01	8.943820196347524D-01
	2.814576927076331D 00	5.954641306249837D-02	9.935982868733539D-01
	4.404042837520693D 00	1.327594473686868D-02	1.085717750794843D 00
	6.382286326243499D 00	1.989242558825718D-03	1.176195397294681D 00
	8.779063402427019D 00	1.953788739313358D-04	1.269335406067786D 00
	1.163526898203052D 01	1.211783690157276D-05	1.369487956109202D 00
	1.500803773429226D 01	4.497403096244903D-07	1.482073652684251D 00
	1.897984650521722D 01	9.233288448468889D-09	1.615098477518938D 00
	2.367626946970571D 01	9.300605842548522D-11	1.782314836142612D 00
	2.930543203250322D 01	3.770187028212009D-13	2.011640248695934D 00
	3.626565595370499D 01	4.223848866124882D-16	2.375104013686701D 00
	4.557681378574562D 01	5.096324515687043D-20	3.169682808838167D 00
16	0.0000000000000000D 00	1.938563815408533D-01	1.938563815408533D-01
	1.800710643375981D-01	5.000658729586037D-01	5.987300926563404D-01
	6.756712689084677D-01	3.791528549432700D-01	7.451687508059863D-01
	1.490027573064593D 00	1.926238794877799D-01	8.547141192910444D-01
	2.631576330024809D 00	6.822847766340409D-02	9.480795087872316D-01
	4.112778856024546D 00	1.691525340189793D-02	1.033797959571884D 00
	5.950906773289312D 00	2.907524305254150D-03	1.116784371788462D 00
	8.169299706816339D 00	3.400521416723136D-04	1.200680058591036D 00
	1.079932394167478D 01	2.630997288377490D-05	1.288864306587349D 00
	1.388349802967013D 01	1.294116582584246D-06	1.385160141322894D 00
	1.748070806823256D 01	3.826203414516876D-08	1.494662397470294D 00
	2.167548985119488D 01	6.271205017311229D-10	1.625161905873936D 00
	2.659611595417350D 01	5.039291674734248D-12	1.790259121023334D 00
	3.245477891680551D 01	1.621573537730887D-14	2.017753194246388D 00
	3.965724995094955D 01	1.424202563882194D-17	2.379559586429555D 00
	4.924250371563341D 01	1.305115116541583D-21	3.172462218749649D 00

S=-2/3

N	A	H	H X EXP(A)
14	0.0000000000000000D 00	1.000509836360825D 00	1.000509836360825D 00
	1.542304053642602D-01	9.887557968764793D-01	1.153640400623408D 00
	6.683363699048020D-01	4.678000124728797D-01	9.126726296279853D-01
	1.552795350753598D 00	1.694152664386372D-01	8.004293326797844D-01
	2.820166270659195D 00	4.372619324462246D-02	7.337097987965822D-01
	4.489503813459974D 00	7.749517343418696D-03	6.903055851762986D-01
	6.588157973473498D 00	9.107209849055800D-04	6.615855294451632D-01
	9.154687389103898D 00	6.805533767233680D-05	6.437131925036661D-01
	1.224380241962062D 01	3.058322984667409D-06	6.351834546607893D-01
	1.593523436195428D 01	7.637936603963801D-08	6.361512729889065D-01
	2.035107519811543D 01	9.412543517331875D-10	6.487339949819672D-01
	2.569433401033817D 01	4.708247656298454D-12	6.788389855118249D-01
	3.235411914587151D 01	6.607386948020336D-15	7.434386799481931D-01
	4.132689062471407D 01	1.032051435356064D-18	9.156761334772813D-01
15	0.0000000000000000D 00	9.772421657477831D-01	9.772421657477831D-01
	1.437006352132780D-01	9.758229540337647D-01	1.126625099975961D 00
	6.224475928489171D-01	4.780299065580490D-01	8.908008498423223D-01
	1.445122465764437D 00	1.839700180964524D-01	7.804691903807234D-01
	2.621784774640996D 00	5.191313739736870D-02	7.143383015527538D-01
	4.167527559171994D 00	1.038872669005193D-02	6.706509482582995D-01
	6.103687759877365D 00	1.432074838396393D-03	6.408606819847160D-01
	8.459719040427753D 00	1.315520106646191D-04	6.210217175541304D-01
	1.127621642010367D 01	7.720442000674881D-06	6.093134048926229D-01
	1.460999693062290D 01	2.734487995161484D-07	6.052252920708831D-01
	1.854317066961785D 01	5.391814299626746D-09	6.094409397328389D-01
	2.320083873702439D 01	5.241227105096944D-11	6.243515832224276D-01
	2.879042224108773D 01	2.057527965500591D-13	6.559447459125748D-01
	3.570896347069692D 01	2.237252455856863D-16	7.209759588590843D-01
	4.497306836956846D 01	2.620534458796950D-20	8.911386950110205D-01
16	0.0000000000000000D 00	9.559977708402226D-01	9.559977708402226D-01
	1.345175337981963D-01	9.632862279980015D-01	1.101984723141795D 00
	5.824730217169906D-01	4.864231739778132D-01	8.709216294220870D-01
	1.351514123095690D 00	1.973555758971163D-01	7.624379885285833D-01
	2.449829192337442D 00	6.015645888547950D-02	6.969948407380007D-01
	3.889591858798265D 00	1.336172714930630D-02	6.532672406663443D-01
	5.687805255921033D 00	2.109512246394792D-03	6.228227786295397D-01
	7.867556826120563D 00	2.304258421214569D-04	6.016829417036344D-01
	1.045996745629979D 01	1.684900055655584D-05	5.878686380162935D-01
	1.350730441234813D 01	7.900962287151857D-07	5.805341342811414D-01
	1.706817875618150D 01	2.241738519429360D-08	5.796966681541007D-01
	2.122679862309957D 01	3.543759965433396D-10	5.863349848472840D-01
	2.611100231109778D 01	2.757055997326914D-12	6.029895803846641D-01
	3.193232222607752D 01	8.613811531947218D-15	6.356626599876156D-01
	3.909531372146495D 01	7.358023616723696D-18	7.008743961642336D-01
	4.863582468511255D 01	6.557393482986895D-22	8.689634871662317D-01

TABLE I (continued)

S= 0.0

N	A	H	H X EXP(A)
2	0.0000000000000000D 00	5.0000000000000000D-01	5.0000000000000000D-01
	2.0000000000000000D 00	5.0000000000000000D-01	3.6945280494653240D 00
3	0.0000000000000000D 00	3.3333333333333336D-01	3.3333333333333336D-01
	1.2679491924311230D 00	6.2200846792814620D-01	2.2103428077469140D 00
	4.7320508075688770D 00	4.4658198738520470D-02	5.0699626028693990D 00
4	0.0000000000000000D 00	2.5000000000000002D-01	2.5000000000000002D-01
	9.3582227524087800D-01	6.2905268086775280D-01	1.6036494771406650D 00
	3.3054072893322790D 00	1.1835638545510060D-01	3.2263526828385830D 00
	7.7587704831436330D 00	2.5909336771469500D-03	6.0680269816134860D 00
5	0.0000000000000000D 00	2.0000000000000002D-01	2.0000000000000002D-01
	7.4329192798143140D-01	6.0120469010385960D-01	1.2642412098356090D 00
	2.5716350076462780D 00	1.8573233407684510D-01	2.4307170638511680D 00
	5.7311787516890990D 00	1.2942849620453810D-02	3.9907012027515120D 00
	1.0953894312683190D 01	1.2012619884232950D-04	6.8683685390843840D 00
6	0.0000000000000000D 00	1.6666666666666670D-01	1.6666666666666670D-01
	6.1703085327827030D-01	5.6401481088726140D-01	1.0453545246162800D 00
	2.1129659585785240D 00	2.3771356660681740D-01	1.9665428975738340D 00
	4.6108331510175320D 00	3.0561921214471850D-02	3.0735483279662330D 00
	8.3990669712048410D 00	1.0381978207811750D-03	4.6126292820188800D 00
	1.4260103065920830D 01	4.8368040025232790D-06	7.5447101892458020D 00
7	0.0000000000000000D 00	1.4285714285714310D-01	1.4285714285714310D-01
	5.2766812171112880D-01	5.2618327809818770D-01	8.9186761796024620D-01
	1.7962998096434090D 00	2.7495999298046100D-01	1.6572674601059890D 00
	3.8766415204769120D 00	5.2366022411779790D-02	2.5272814396752240D 00
	6.9188165667047210D 00	3.5654681882885950D-03	3.6051257854831340D 00
	1.1234610429083110D 01	6.7918934286162420D-05	5.1418591802015210D 00
	1.7645963552380710D 01	1.7652985501646250D-07	8.1350975153767690D 00
8	0.0000000000000000D 00	1.2500000000000003D-01	1.2500000000000003D-01
	4.6102421980499480D-01	4.9066965826409800D-01	7.7805353075785770D-01
	1.5635861896542630D 00	3.0040474317896620D-01	1.4347083956867780D 00
	3.3520505025367340D 00	7.5512739453312290D-02	2.1567373638154500D 00
	5.9162972490205390D 00	8.0799031678388080D-03	2.9979294238359220D 00
	9.4206993830215890D 00	3.2911142049742040D-04	4.0616332969862000D 00
	1.4194165548007460D 01	3.8385256751540480D-06	5.6054724520957310D 00
	2.1092176907954410D 01	5.9896135285108520D-09	8.6619334658240370D 00
9	0.0000000000000000D 00	1.1111111111111114D-01	1.1111111111111114D-01
	4.0938357320318510D-01	4.5832943402591850D-01	6.9019335779276190D-01
	1.3849631848031400D 00	3.1696536452955730D-01	1.2661748338184850D 00
	2.9562545561688620D 00	9.8095771277941840D-02	1.8859723747882690D 00
	5.1819431010400710D 00	1.4500619391520490D-02	2.5815221026166270D 00
	8.1617096881458170D 00	9.7193442940378570D-04	3.4058259260529790D 00
	1.2070055126837150D 01	2.5570897031535080D-05	4.4637956154705390D 00
	1.7249735526148990D 01	1.9414548151885540D-07	6.0199410804020730D 00
	2.4585955243652780D 01	1.9203617579236110D-10	9.1396116690556360D 00

S=-0.5

N	A	H	H X EXP(A)
2	0.0000000000000000D 00	1.1816359006036770D 00	1.1816359006036770D 00
	1.5000000000000000D 00	5.9081795030183860D-01	2.6478623504272870D 00
3	0.0000000000000000D 00	9.4530872048294250D-01	9.4530872048294250D-01
	9.1886116991581030D-01	7.8723864630448270D-01	1.9731619935028570D 00
	4.0811388300841900D 00	3.9906484118091830D-02	2.3629772510719750D 00
4	0.0000000000000000D 00	8.1026461755680780D-01	8.1026461755680780D-01
	6.6632590770237070D-01	8.5121450522025610D-01	1.6573746065672790D 00
	2.8007750541502570D 00	1.0903116563825420D-01	1.7943692004503690D 00
	7.0328990381473720D 00	1.9435624901990400D-03	2.2026614592206460D 00
5	0.0000000000000000D 00	7.2023521560605190D-01	7.2023521560605190D-01
	5.2352607673826900D-01	8.6530311800511200D-01	1.4606049054901850D 00
	2.1566487632690940D 00	1.7694905478875320D-01	1.5292162501891010D 00
	5.1373875461767110D 00	9.8872485510739050D-03	1.6835054029573410D 00
	1.0182437613815920D 01	7.9213954526528890D-05	2.0940071619533670D 00
6	0.0000000000000000D 00	6.5475928691459270D-01	6.5475928691459270D-01
	4.3139880714785140D-01	8.5871950471225120D-01	1.3219208388819230D 00
	1.7597536984236960D 00	2.3445575033541730D-01	1.3624237621333410D 00
	4.1044653628283150D 00	2.3822790889823090D-02	1.4439072494567810D 00
	7.7467037795425570D 00	6.9363893264669190D-04	1.6050337377020620D 00
	1.3457678352057580D 01	2.8791207874285170D-06	2.0130535723447780D 00
7	0.0000000000000000D 00	6.0439318792116270D-01	6.0439318792116270D-01
	3.6694987730837070D-01	8.4323259379708790D-01	1.2170591674066430D 00
	1.4885342923104520D 00	2.8064664137404720D-01	1.2434321105735280D 00
	3.4340379684240710D 00	4.1725550592339920D-02	1.2935189266316850D 00
	6.3490679256803790D 00	2.4149199854387770D-03	1.3812360696756920D 00
	1.0540469858448340D 01	4.0860720805414190D-05	1.5451616467034970D 00
	1.6820970077828380D 01	9.6514637001462780D-08	1.9491607912798210D 00
8	0.0000000000000000D 00	5.6410030872641900D-01	5.6410030872641900D-01
	3.1930363392063000D-01	8.2405737499779840D-01	1.1340423068932090D 00
	1.2907586229591530D 00	3.1697783159187210D-01	1.1523867005670010D 00
	2.9583744586966500D 00	6.1560067745092320D-02	1.1860548995284220D 00
	5.4090315972444330D 00	5.5561376858255640D-03	1.2413252070540770D 00
	8.8040795780567750D 00	2.0000882465000040D-04	1.3323320102180910D 00
	1.3468535743251480D 01	2.1182310954221370D-06	1.4972147320338150D 00
	2.0249916365870880D 01	3.0449516085070350D-09	1.8967379416552100D 00
9	0.0000000000000000D 00	5.3091793762486460D-01	5.3091793762486460D-01
	2.8263364811659910D-01	8.0365293894082510D-01	1.0661413091471960D 00
	1.1398738015816140D 00	3.4529659534019500D-01	1.0795262278169990D 00
	2.6015248434060290D 00	8.1840068299512710D-02	1.1035547061563330D 00
	4.7241145375277900D 00	1.0134699915255100D-02	1.1414785882489130D 00
	7.6052562992316140D 00	5.9728657339550720D-04	1.1997854653355500D 00
	1.1417182076545830D 01	1.4224578280042660D-05	1.2925834004257580D 00
	1.6499410797655810D 01	9.9541579632616130D-08	1.4574967411742070D 00
	2.3730003995934710D 01	9.1611578615972770D-11	1.8525082799790310D 00

Table I (continued)

S= 0.0

N	A	H	H X EXP(A)
10	0.0000000000000000D 00	1.00000000000003D-01	1.00000000000003D-01
	3.68178452941741 5D-01	4.29223715397022 1D-01	6.20271147465299 7D-01
	1.24335796214048 2D 00	3.27003640010177 1D-01	1.13379905057347 8D 00
	2.64603384138420 7D 00	1.19039476645666 5D-01	1.67822004884611 5D 00
	4.61688251463506 0D 00	2.24774010650445 3D-02	2.27422115197976 9D 00
	7.22178653939657 0D 00	2.15703380864260 4D-03	2.95283374740049 3D 00
	1.05673208077418 6D 01	9.69850341914401 1D-05	3.76733151428377 9D 00
	1.48359145152609 3D 01	1.73903747966685 9D-06	4.82463818715638 3D 00
	2.03821819854492 5D 01	8.99589431338315 2D-09	6.39607444590997 2D 00
	2.81183433810498 9D 01	5.88345278223197 1D-12	9.57797376239765 6D 00
11	0.0000000000000000D 00	9.09090909090912 0D-02	9.09090909090912 0D-02
	3.34528676324752 5D-01	4.03124049073933 4D-01	5.63277839698470 7D-01
	1.12825335587663 7D 00	3.32305097212983 7D-01	1.02690722540239 7D 00
	2.39586992474730 6D 00	1.37840175459044 9D-01	1.51317412015161 3D 00
	4.16684098792876 7D 00	3.15656117001687 0D-02	2.03633983870247 8D 00
	6.48735303138081 1D 00	3.98345274054197 3D-03	2.61626235811058 0D 00
	9.42835481333560 6D 00	2.64059408580440 5D-04	3.28385607894124 0D 00
	1.31017235803678 0D 01	8.35679945123850 9D-06	4.09304231335267 7D 00
	1.76964875668462 2D 01	1.06307842563414 2D-07	5.15290580826100 8D 00
	2.35777870883601 5D 01	3.88191011916217 6D-10	6.74137216603568 7D 00
	3.16828009748319 4D 01	1.73636904784675 9D-13	9.98407121705042 7D 00
12	0.0000000000000000D 00	8.33333333333337 1D-02	8.33333333333337 1D-02
	3.06526702130052 1D-01	3.79714605707903 5D-01	5.15917379322680 2D-01
	1.03279739877972 5D 00	3.34185024695849 5D-01	9.38696494136748 2D-01
	2.18961194196838 8D 00	1.54345614942189 4D-01	1.37857589156814 9D 00
	3.79904760605496 4D 00	4.13325991609596 1D-02	1.84585731993463 0D 00
	5.89491417150274 5D 00	6.48178996203448 6D-03	2.35408612175107 8D 00
	8.52729200807027 0D 00	5.78704289833127 6D-04	2.92289063919429 4D 00
	1.17710065654906 9D 01	2.76826304964637 4D-05	3.58336092597244 2D 00
	1.57422602870261 9D 01	6.39308282457391 6D-07	4.39022831311625 7D 00
	2.06358056686411 3D 01	5.95332648256056 0D-09	5.45476734072129 0D 00
	2.68263499493707 0D 01	1.57887123236530 3D-11	7.06126425392429 9D 00
	3.52743907009651 3D 01	4.96613250427848 4D-15	1.03631440972492 6D 01
13	0.0000000000000000D 00	7.69230769230773 2D-02	7.69230769230773 2D-02
	2.82858348239914 1D-01	3.58673400650497 2D-01	4.75929900326684 5D-01
	9.52326041364613 4D-01	3.33603691974734 7D-01	8.64611072678860 9D-01
	2.01649213857768 1D 00	1.68602825109083 5D-01	1.26653225923596 8D 00
	3.49235406977802 5D 00	5.14073303714625 0D-02	1.68941035775682 4D 00
	5.40549102001572 2D 00	9.62575626204208 7D-03	2.14293887031611 5D 00
	7.79281394041241 5D 00	1.08967280634439 5D-03	2.64041508758132 9D 00
	1.07073886889890 9D 01	7.16454608645599 2D-05	3.20146312472386 7D 00
	1.42271523637899 8D 01	2.55582356005338 3D-06	3.85748844676654 2D 00
	1.84719966342300 0D 01	4.43079715196184 2D-08	4.66410239041936 4D 00
	2.36417837524008 9D 01	3.09754719741042 3D-10	5.73474562136307 6D 00
	3.01200586261063 4D 01	6.10789792031400 5D-13	7.35981717403427 3D 00
	3.88892843760953 2D 01	1.38280470020063 5D-16	1.07192020399042 3D 01

S=-0.5

N	A	H	H X EXP(A)
10	0.0000000000000000D 00	5.02974888276188 3D-01	5.02974888276188 3D-01
	2.53532554974419 0D-01	7.83217977226062 4D-01	1.00923066270447 7D 00
	1.02084427772039 0D 00	3.67265402613995 1D-01	1.01935875019652 7D 00
	2.32309607702246 6D 00	1.01620773818104 4D-01	1.03726638740054 8D 00
	4.19935060065729 3D 00	1.59777335554460 0D-02	1.06480472110310 3D 00
	6.71397431661502 8D 00	1.34175504281436 6D-03	1.10538924193429 1D 00
	9.97200915953934 8D 00	5.44183955263233 5D-05	1.16555905952566 1D 00
	1.41540536712780 4D 01	8.97648629444627 2D-07	1.25931326527366 3D 00
	1.96119028191659 5D 01	4.32610201972711 8D-09	1.42376374867425 0D 00
	2.72512365230270 6D 01	2.65259418899704 2D-12	1.81439759218570 5D 00
11	0.0000000000000000D 00	4.79023703120179 2D-01	4.79023703120179 2D-01
	2.29872980518656 2D-01	7.63338147227006 0D-01	9.60615375063159 7D-01
	9.24481546986657 3D-01	3.84240648133996 4D-01	9.68502041880489 8D-01
	2.09941046270879 8D 00	1.20359293317824 8D-01	9.82295169422618 7D-01
	3.78288087370729 0D 00	2.28281316748688 4D-02	1.00312408532758 2D 00
	6.01991802770146 1D 00	2.50996408345282 5D-03	1.03296281498506 2D 00
	8.88034759799670 8D 00	1.49567977346201 6D-04	1.07528745677709 0D 00
	1.24748324048362 0D 01	4.34376979611335 2D-06	1.13662623836177 8D 00
	1.69908472935425 5D 01	5.14246036011864 4D-08	1.23084162555947 9D 00
	2.27910028949489 4D 01	1.76372224840999 5D-10	1.39455930454943 2D 00
	3.08064059170527 2D 01	7.44073014027212 8D-14	1.78101530240309 1D 00
12	0.0000000000000000D 00	4.58196585593214 9D-01	4.58196585593214 9D-01
	2.10257418483179 8D-01	7.44287649755131 8D-01	9.18447738795949 5D-01
	8.44839416425212 4D-01	3.97289797156046 3D-01	9.24733324827631 9D-01
	1.91556557473617 3D 00	1.37780578858175 0D-01	9.35637379374370 6D-01
	3.44353726814079 5D 00	3.04141680089683 6D-02	9.51886515150156 9D-01
	5.46164477269441 8D 00	4.13913574992128 8D-03	9.74701999441431 2D-01
	8.01911268878234 5D 00	3.31123398283749 2D-04	1.00611184275937 8D 00
	1.11898757161764 4D 01	1.44985918360045 7D-05	1.04960792341492 3D 00
	1.50891282200211 5D 01	3.11067865829154 5D-07	1.11168122831536 0D 00
	1.99102576427684 7D 01	2.71925930080578 7D-09	1.20605093631684 9D 00
	2.60256553818914 2D 01	6.81662819606110 7D-12	1.36888866355457 0D 00
	3.43901258998806 9D 01	2.03207692412735 3D-15	1.75138672833620 6D 00
13	0.0000000000000000D 00	4.39868722169486 7D-01	4.39868722169486 7D-01
	1.93729644504762 2D-01	7.26177978551783 5D-01	8.81411657824116 9D-01
	7.77893582258528 9D-01	4.07242273356249 0D-01	8.86518378503704 2D-01
	1.76167406209561 9D 00	1.53777990351618 1D-01	8.95322513174800 6D-01
	3.16128799814416 0D 00	3.84861979308179 1D-02	9.08311771055236 9D-01
	5.00157499906472 1D 00	6.23141744025128 5D-03	9.26282093149992 8D-01
	7.31875758565791 7D 00	6.30167277490970 0D-04	9.50499476004379 9D-01
	1.01652245251148 8D 01	3.78319459146811 7D-05	9.83013763775175 3D-01
	1.36181877122672 6D 01	1.25140649993822 5D-06	1.02731157954913 3D 00
	1.77966656442674 9D 01	2.03407650060369 8D-08	1.08983554438880 1D 00
	2.28992910181960 2D 01	1.34392768354124 5D-10	1.18416338617301 5D 00
	2.93074585872096 9D 01	2.51762997549309 7D-13	1.34604458044411 7D 00
	3.79982546412189 3D 01	5.42384702807686 4D-17	1.72480397746206 4D 00

For $s = 0$, comparison with the 8-place tables of [8] discloses that 17 of 240 entries therein are incorrect in the 8th digit, presumably due to rounding, and 4 cases (in their notation: A_2 of $n = 6$, A_3 of $n = 12$, A_{12} of $n = 14$, X_{15} of $n = 15$) of misprints or errors. Among the 24 possible comparisons with $s = 1$ of [1], there are 5 cases in which, rounding their a's to 16 figures, our last digits are in error by 1 unit; there are 4 cases in which their weights and our $a_j \cdot H_j$ differ by one when rounded to 14 digits. Since reference [5] is known to the author only via Mathematical Reviews, comparison with it cannot be made.

3. The Error Term. Since published formulas do not include that for the error term, this term is derived below, some of the essential ideas being found in [6].

If $f(x)$ is a polynomial of degree $2n - 2$ or less, it can be written

$$f(x) = \Lambda_n^s(x) \cdot \sum_{r=0}^{n-2} A_r \Lambda_r^s(x) + \sum_{k=1}^{n} B_k \frac{\Lambda_n^s(x)}{x - a_k} \tag{3}$$

where $\Lambda_0^s(x) = 1$ and the B's are to be evaluated from

$$f(a_j) = B_j \left[\frac{d}{dx} \Lambda_n^s(x) \right]_{x=a_j} \qquad j = 1, \cdots, n \tag{4}$$

and the A's are to be determined from the requirements

$$\begin{aligned} f'(a_j) = \left[\frac{d}{dx} \Lambda_n^s(x) \right]_{x=a_j} \cdot \sum_{r=0}^{n-2} A_r \Lambda_r^s(a_j) + \sum_{k=1; k \neq j}^{n} (B_k + B_j) \\ \cdot \left. \frac{\Lambda_n^s(x)}{(x - a_j)(x - a_k)} \right|_{x=a_j} \qquad (j = 2, \cdots, n). \end{aligned} \tag{5}$$

Thus the left and right members of (3) are polynomials of degree $2n - 2$ or less coinciding at the n points $x = a_j$ and with equal derivatives at $n - 1$ points $x = a_j \neq 0$, hence are identical.

If $f(x)$ is not a polynomial of degree $2n - 2$ or less, then (3) will not be exact, but must contain a remainder term:

$$f(x) = \Lambda_n^s(x) \cdot \sum_{r=0}^{n-2} A_r \Lambda_r^s(x) + \sum_{k=1}^{n} B_k \frac{\Lambda_n^s(x)}{x - a_k} + R_{2n-1}(x) \tag{6}$$

and some form for $R_{2n-1}(x)$ is to be sought. We examine the possibility that

$$R_{2n-1}(x) = K(x) \cdot \frac{[\Lambda_n^s(x)]^2}{x}. \tag{7}$$

Let the right member of (3) be denoted by $\theta(x)$, and consider the function

$$\Phi(x) = f(x) - \theta(x) - R_{2n-1}(x).$$

With A's and B's determined by (4) and (5), and with R in the form (7), $\Phi(x)$ has zeros at $x = a_j$ and vanishing first derivative at $x = a_j \neq 0$. Then, selecting some arbitrary $x = \bar{x}$ different from all a_j, $K(\bar{x})$ can be determined so that $\Phi(\bar{x}) = 0$. Then $\Phi(x)$ has double zeros at $x = a_j$ $(j = 2, \cdots, n)$ and simple zeros at $x = 0$ and $x = \bar{x}$. Hence, by $(2n - 1)$-fold application of Rolle's Theorem,

$$\Phi^{(2n-1)}(\xi) = f^{(2n-1)}(\xi) - K(\bar{x}) \cdot (2n - 1)! = 0 \quad \text{for some } \xi.$$

This permits writing the remainder term in the form

$$R_{2n-1}(x) = \frac{f^{(2n-1)}(\xi)}{(2n-1)!} \frac{[\Lambda_n^s(x)]^2}{x}, \qquad 0 \leqq \xi \leqq \max(a_j, \bar{x}).$$

When (6) is substituted into the left member of (1), the terms of the first summation vanish because they are orthogonal with respect to the weight function; the terms of the second summation become the nonremainder terms of the right member; the integral of $R_{2n-1}(x)$ becomes E as follows:

$$\begin{aligned} E &= \int_0^\infty x^s e^{-x} \frac{f^{(2n-1)}(\xi)}{(2n-1)!} \frac{[\Lambda_n^s(x)]^2}{x} dx, \qquad 0 \leqq \xi \leqq \max(a_j, x), \\ &= \frac{f^{(2n-1)}(\eta)}{(2n-1)!} \int_0^\infty x^{s+1} e^{-x} \frac{[\Lambda_n^s(x)]^2}{x} dx, \qquad 0 \leqq \eta \leqq \infty, \\ &= \frac{f^{(2n-1)}(\eta)}{(2n-1)!} \int_0^\infty x^{s+1} e^{-x} [\Lambda_{n-1}^{s+1}(x)]^2 dx, \qquad 0 \leqq \eta \leqq \infty, \\ &= \frac{f^{(2n-1)}(\eta)}{(2n-1)!} \cdot (n-1)!\, \Gamma(n+s+1), \qquad 0 \leqq \eta \leqq \infty. \end{aligned}$$

4. Acknowledgment. The author wishes to thank Mr. Hugh Hopper for his interest in the problem and for his careful programming of equations for machine computing of Table I.

Computer Department
General Electric Company
Huntsville, Alabama

1. Philip Rabinowitz & George Weiss, "Tables of abscissas and weights for numerical evaluation of integrals of the form $\int_0^\infty e^{-x}x^n f(x)\, dx$," *MTAC*, v. 13, 1959, pp. 285–294. MR **21** #6713.

2. P. Concus, D. Cassatt, G. Jaehnig & E. Melby, "Tables for the evaluation of $\int_0^\infty x^\beta e^{-x} f(x)\, dx$ by Gauss-Laguerre quadrature," *Math. Comp.*, v. 17, 1963, pp. 245–256. MR **28** #1757.

3. Paul Concus, "Additional tables for the evaluation of $\int_0^\infty x^\beta e^{-x} f(x)\, dx$ by Gauss Laguerre quadrature," *Math. Comp.*, v. 18, 1964, pp. 523.

4. T. S. Shao, T. C. Chen & R. M. Frank, "Tables of zeros and Gaussian weights of certain associated Laguerre polynomials and the related generalized Hermite polynomials," *Math. Comp.*, v. 18, 1964, pp. 598–616.

5. V. S. Aĭzenštat, V. I. Krylov & A. S. Metel'skiĭ, a) *Tables of Laguerre Polynomials and Functions*, Izdat. Akad. Nauk BSSR, Minsk, 1963. MR **27** #1627.
b) *Tables for the Numerical Laplace Transform and the Evaluation of Integrals of the Form* $\int_0^\infty x^s e^{-x} f(x)\, dx$, Izdat. Akad. Nauk BSSR, Minsk, 1962. MR **27** #1628.

6. D. Burnett, "The numerical calculation of $\int_0^\infty x^m e^{-x} f(x)\, dx$," *Proc. Cambridge Philos. Soc.*, v. 33, 1937, pp. 359–362.

7. Zdeněk Kopal, *Numerical Analysis: with Emphasis on the Application of Numerical Techniques to Problems of Infinitesimal Calculus in Single Variable*, Wiley, New York, 1955; 2nd ed., 1961. MR **17**, 1007; MR **25** #733.

8. V. I. Krylov & N. P. Fedenko, "Approximate representation of the integral $\int_0^\infty x^s e^{-x} f(x)\, dx$ by mechanical quadrature containing the value $f(0)$," (Russian) *Vesci Akad. Navuk BSSR Ser. Fiz.-Tèhn. Navuk*, v. 1962, no. 2, pp. 5–9. MR **26** #5713.

9. C. R. Cassity & H. R. Hopper, *Abscissas and Coefficients for the Generalized Gauss-Laguerre Quadrature Formula using the Zero Ordinate*, NASA Report TMX-53099, National Aeronautics and Space Administration, George C. Marshall Space Flight Center, MS-IPL, Huntsville, Alabama.

10. Henri Mineur, *Techniques de Calcul Numérique à l'Usage des Mathématiciens, Astronomes, Physiciens et Ingénieurs*, Béranger, Paris, 1952 (Note by Jean Bouzitat, pp. 557–605). MR **15**, 557.