and take

$$X_0 = \frac{1}{2} A^* = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

Here, formula (17) is used to obtain:

$$X_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}\left\{2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}\right\}$$

$$= \frac{1}{4}\begin{pmatrix} 2 & 1 \\ 1 & 2 \\ -1 & 1 \end{pmatrix},$$

$$X_2 = \frac{1}{16}\begin{pmatrix} 10 & 5 \\ 5 & 10 \\ -5 & 5 \end{pmatrix},$$

$$X_3 = \frac{1}{256}\begin{pmatrix} 170 & 85 \\ 85 & 170 \\ -85 & 85 \end{pmatrix}, \qquad \text{etc.,}$$

converging to:

$$A^+ = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \\ -1 & 1 \end{pmatrix}.$$

Technion, Israel Institute of Technology
Haifa, Israel

1. G. Schulz, "Iterative Berechnung der reziproken Matrix," *Z. Angew. Math. Mech.* v. 13, 1933, pp. 57–59.
2. R. Penrose, "A generalized inverse for matrices," *Proc. Cambridge Philos. Soc.*, v. 51, 1955, pp. 406–413. MR **16**, 1082.
3. A. Ben-Israel & A. Charnes, "Contributions to the theory of generalized inverses," *J. Soc. Indust. Appl. Math.*, v. 11, 1963, pp. 667–699.
4. W. Dück, "Fehlerabschätzungen für das Iterationsverfahren von Schulz zur Bestimmung der Inversen einer Matrix," *Z. Angew. Math. Mech.*, v. 40, 1960, pp. 192–194. MR **22** #3102.
5. A. S. Householder, *Theory of Matrices in Numerical Analysis*, Blaisdell, New York, 1964.

A Note on the Maximum Value of Determinants over the Complex Field

By C. H. Yang

The purpose of this note is to extend a theorem on determinants over the real field to the corresponding theorem over the complex field.

Theorem. *Let $D(n)$ be an nth order determinant with complex numbers as its entries. Then*

$$\text{(1)} \qquad \operatorname*{Max}_{|a_{jk}| \leq K} |D(n)| = \operatorname*{Max}_{|a_{jk}| = K} |D(n)|.$$

Received June 5, 1964. Revised December 8, 1964.

In other words, $D(n)$ is a function of n^2 variables a_{jk} which vary over the bounded and closed domain $\bar{D}$: $\{| a_{jk} | \leqq K\}$; hence this function is bounded and attains its maximum value on the boundary of the domain $\bar{D}$.

Proof. Let $a_{jk} = r_{jk}e^{i\theta_{jk}}$ and $A_{jk} = R_{jk}e^{i\phi_{jk}}$ = the co-factor of a_{jk}, where $K \geqq r_{jk} \geqq 0$ and $R_{jk} \geqq 0$. Then, expanding by the jth row, we have

$$
\begin{aligned}
| D(n) | &= \left| \sum_{k=1}^{n} a_{jk}A_{jk} \right| = \left| \sum_{k=1}^{n} r_{jk}R_{jk}\, e^{i(\theta_{jk}+\phi_{jk})} \right| \\
&\leqq \sum_{k=1}^{n} r_{jk}R_{jk} \leqq \sum_{k=1}^{n} KR_{jk} = D'(n),
\end{aligned} \tag{2}
$$

where $D'(n)$ is the nth order determinant whose entries are

$$
a'_{jk} = \begin{cases} a_{jk}, & \text{if } r_{jk} = K \text{ and } \theta_{jk} + \phi_{jk} \equiv 0 \pmod{2\pi}, \\ Ke^{-i\phi_{jk}}, & \text{if } r_{jk} < K \text{ or } \theta_{jk} + \phi_{jk} \not\equiv 0 \pmod{2\pi}. \end{cases} \tag{3}
$$

By applying the same process to the other rows, we obtain a determinant $D^*(n)$ whose entries $| a^*_{jk} | = K$ and $| D^*(n)| \geqq | D(n)|$. Hence, $\operatorname{Max}_{|a_{jk}| \leqq K} | D(n)| \leqq \operatorname{Max}_{|a_{jk}| = K} | D(n)|$; thus the proof of the theorem can be completed since the reverse inequality is trivial.

50 Mohigan Drive
Oneonta, New York

On the Numerical Solution of $y' = f(x, y)$ by a Class of Formulae Based on Rational Approximation

By John D. Lambert and Brian Shaw

1. Introduction. Most finite difference formulae in common usage for the numerical solution of first-order differential equations are based on polynomial approximation. Two exceptions are the formulae based on exponential approximation proposed by Brock and Murray [1], and the formulae of Gautschi [2] which are derived from trigonometric polynomials. The use of rational functions as approximants has been studied by many authors, including Remes [3], Maehly [4] and Stoer [5], but the main concern of most of this work has been the direct approximation of a given function. Algorithms for interpolation based on rational functions have been proposed by Wynn [6], and methods for numerical integration and differentiation based on Padé approximation have been studied by Kopal [7]. It is the purpose of the present paper to derive a class of formulae, based on rational approximation, for the numerical solution of the initial value problem

$$
y' = f(x, y), \qquad y(x_0) = y_0 . \tag{1}
$$

The formulae proposed give exact results when the theoretical solution of (1) is a rational function of a certain degree, just as many of the classical difference formulae

Received December 10, 1964.