the values of N_k, $0 \leqq k \leqq e$, and the number of occurrences of the various partitions of e into which the solutions are grouped.

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Tables of Values of Three Infinite Integrals

By Chih-Bing Ling and Hsien-Chueh Wu

Sometime ago, the senior author [1] evaluated the following two integrals to five decimal places for integral values of m and p up to 15 and 8, respectively.

$$(1) \qquad \begin{aligned} I(m, p) &= \frac{p^{m+1}}{2^p(m!)} \int_0^\infty \frac{x^m\,dx}{\sinh^p x} \qquad (m \geqq p \geqq 1) \\ J(m, p) &= \frac{p^{m+1}}{2^p(m!)} \int_0^\infty \frac{x^m\,dx}{\cosh^p x} \qquad (m \geqq 0, p \geqq 1) \end{aligned}$$

Two particular integrals $I(m, m)$ and $I(m, m-1)$ were further evaluated by Nelson and the senior author [2] to seven decimal places for $m = 1(1)40$. Nelson also evaluated these two integrals for the same range of values of m to 12D and 18D, respectively, in two papers [3], [4].

In the present paper, the two preceding integrals will be evaluated to 8D for m and p up to 25 and 12, respectively. The same method of evaluation will be used. The various sums of inverse powers required in the computation were tabulated to 32D by Glaisher [5], [6], and also appear in two well-known mathematical tables [7], [8]. The results are shown in Tables 1 and 2. Table 3 shows the factor $2^p(m!)/p^{m+1}$, also to 8D.

In addition, the following integral will also be evaluated to 8D for the same range of values of m and p.

$$(2) \qquad S(m, p) = \int_0^\infty \frac{\sin^m x}{x^p}\,dx \qquad (m \geqq p \geqq 1).$$

The integers m and p are restricted as indicated, and $S(2m, 1)$ is to be excluded on account of its divergence. This last integral occurs in certain branches of mathematical physics, and on that account it was thought desirable to include a table of its values.

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By repeated integration by parts, it can be shown that, for $m \geqq p \geqq 2$,

$$\int_0^\infty \frac{\sin^m x}{x^p}\,dx = \frac{1}{(p-1)!}\int_0^\infty \left(\frac{d^{p-1}}{dx^{p-1}}\sin^m x\right)\frac{dx}{x}. \tag{3}$$

TABLE 1

Values of $I(m, p)$

m	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$
1	1.23370 06	—	—	—	—	—
2	1.05179 98	1.64493 41	—	—	—	—
3	1.01467 80	1.20205 69	2.21760 30	—	—	—
4	1.00452 38	1.08232 32	1.43600 93	3.00059 11	—	—
5	1.00144 71	1.03692 78	1.20567 08	1.76137 76	4.06672 87	—
6	1.00047 15	1.01734 31	1.10777 40	1.38624 37	2.19850 67	5.51652 79
7	1.00015 52	1.00834 93	1.05951 75	1.21934 84	1.63154 75	2.77832 31
8	1.00005 13	1.00407 74	1.03385 81	1.13200 69	1.37523 87	1.95474 31
9	1.00001 70	1.00200 84	1.01961 14	1.08217 76	1.23745 36	1.58199 14
10	1.00000 57	1.00099 46	1.01149 02	1.05225 59	1.15601 52	1.37965 26
11	1.00000 19	1.00049 42	1.00678 27	1.03369 67	1.10501 10	1.25804 34
12	1.00000 06	1.00024 61	1.00402 40	1.02193 64	1.07184 01	1.18015 47
13	1.00000 02	1.00012 27	1.00239 55	1.01437 51	1.04970 75	1.12811 09
14	1.00000 01	1.00006 12	1.00142 94	1.00946 42	1.03467 31	1.09229 88
15	1.00000 00	1.00003 06	1.00085 44	1.00625 19	1.02432 86	1.06713 23
16	1.00000 00	1.00001 53	1.00051 12	1.00413 99	1.01714 45	1.04917 30
17	1.00000 00	1.00000 76	1.00030 61	1.00274 62	1.01212 09	1.03620 97
18	1.00000 00	1.00000 38	1.00018 34	1.00182 40	1.00859 01	1.02677 16
19	1.00000 00	1.00000 19	1.00010 99	1.00121 27	1.00609 90	1.01985 49
20	1.00000 00	1.00000 10	1.00006 59	1.00080 68	1.00433 63	1.01476 05
21	1.00000 00	1.00000 05	1.00003 95	1.00053 70	1.00308 64	1.01099 37
22	1.00000 00	1.00000 02	1.00002 37	1.00035 76	1.00219 85	1.00820 01
23	1.00000 00	1.00000 01	1.00001 42	1.00023 82	1.00156 70	1.00612 34
24	1.00000 00	1.00000 01	1.00000 85	1.00015 87	1.00111 75	1.00457 67
25	1.00000 00	1.00000 00	1.00000 51	1.00010 58	1.00079 72	1.00342 31

m	$p = 7$	$p = 8$	$p = 9$	$p = 10$	$p = 11$	$p = 12$
7	7.48711 20	—	—	—	—	—
8	3.54366 24	10.16505 0	—	—	—	—
9	2.37486 43	4.55231 39	13.80399 0	—	—	—
10	1.84949 92	2.91780 01	5.88155 91	18.74868 8	—	—
11	1.56430 97	2.19145 90	3.61796 19	7.63443 55	25.46768 7	—
12	1.39189 70	1.79943 04	2.62611 81	4.52067 23	9.94820 28	34.59775 0
13	1.28027 75	1.56261 96	2.09574 34	3.17736 75	5.68541 26	13.00570 9
14	1.20458 38	1.40878 73	1.77698 42	2.46727 67	3.87624 58	7.19016 14
15	1.15154 20	1.30373 79	1.57021 04	2.04384 96	2.93214 41	4.76294 60
16	1.11347 60	1.22938 55	1.42876 09	1.77045 29	2.37461 59	3.51359 30
17	1.08566 81	1.17534 98	1.32819 61	1.58377 11	2.01709 65	2.78382 35
18	1.06507 93	1.13528 78	1.25461 00	1.45096 57	1.77403 31	2.31976 81
19	1.04967 76	1.10512 73	1.19956 77	1.35351 92	1.60148 58	2.00620 60
20	1.03806 40	1.08214 91	1.15768 70	1.28029 06	1.47490 03	1.78452 89
21	1.02925 20	1.06447 81	1.12538 97	1.22422 14	1.37962 64	1.62227 93
22	1.02253 27	1.05078 72	1.10021 49	1.18064 72	1.30645 24	1.50025 29
23	1.01738 94	1.04011 68	1.08042 29	1.14637 61	1.24933 24	1.40646 78
24	1.01344 01	1.03176 09	1.06475 44	1.11915 94	1.20415 37	1.33311 54
25	1.01040 02	1.02519 21	1.05228 01	1.09737 31	1.16803 36	1.27491 87

TABLE 2

Values of J(m, p)

m	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$
0	0.78539 816	0.50000 000	0.29452 431	0.16666 667	0.09203 8847	0.05000 0000
1	0.91596 559	0.69314 718	0.46796 129	0.29543 145	0.17862 567	0.10481 915
2	0.96894 615	0.82246 703	0.61947 444	0.42995 604	0.28207 477	0.17772 033
3	0.98894 455	0.90154 268	0.73891 195	0.55572 132	0.39200 036	0.26354 349
4	0.99615 783	0.94703 283	0.82655 484	0.66435 091	0.49932 579	0.35598 721
5	0.99868 522	0.97211 977	0.88761 860	0.75282 233	0.59761 868	0.44902 072
6	0.99955 451	0.98555 109	0.92856 734	0.82172 292	0.68327 566	0.53776 965
7	0.99984 999	0.99259 382	0.95525 538	0.87355 944	0.75504 738	0.61887 206
8	0.99994 968	0.99623 300	0.97228 188	0.91152 303	0.81332 833	0.69044 362
9	0.99998 316	0.99809 430	0.98297 231	0.93874 818	0.85947 637	0.73182 220
10	0.99999 437	0.99903 951	0.98960 485	0.95795 396	0.89528 182	0.80323 237
11	0.99999 812	0.99951 714	0.99368 336	0.97132 951	0.92261 006	0.84546 155
12	0.99999 937	0.99975 769	0.99617 485	0.98055 174	0.94319 321	0.87959 380
13	0.99999 979	0.99987 854	0.99768 946	0.98686 093	0.95853 101	0.90681 522
14	0.99999 993	0.99993 917	0.99860 693	0.99115 118	0.96986 206	0.92828 614
15	0.99999 998	0.99996 955	0.99916 122	0.99405 496	0.97817 527	0.94506 713
16	0.99999 999	0.99998 476	0.99949 546	0.99601 326	0.98424 064	0.95808 396
17	1.00000 000	0.99999 238	0.99969 672	0.99733 030	0.98864 639	0.96811 839
18	1.00000 000	0.99999 619	0.99981 780	0.99821 418	0.99183 532	0.97581 429
19	1.00000 000	0.99999 809	0.99989 057	0.99880 641	0.99413 704	0.98169 197
20	1.00000 000	0.99999 905	0.99993 430	0.99920 274	0.99579 468	0.98616 563
21	1.00000 000	0.99999 952	0.99996 056	0.99946 772	0.99698 637	0.98956 113
22	1.00000 000	0.99999 976	0.99997 633	0.99964 476	0.99784 188	0.99213 243
23	1.00000 000	0.99999 988	0.99998 579	0.99976 298	0.99845 541	0.99407 599
24	1.00000 000	0.99999 994	0.99999 147	0.99984 189	0.99889 499	0.99554 285
25	1.00000 000	0.99999 997	0.99999 488	0.99989 454	0.99920 974	0.99664 857

m	$p = 7$	$p = 8$	$p = 9$	$p = 10$	$p = 11$	$p = 12$
0	0.02684 4664	0.01428 5714	0.00755 00617	0.00396 82540	0.00207 62670	0.00108 22511
1	0.06017 8400	0.03397 8725	0.01893 5884	0.01044 1832	0.00570 80938	0.00309 77245
2	0.10863 720	0.06487 3818	0.03802 7270	0.02195 6642	0.01252 0112	0.00706 45307
3	0.17077 088	0.10749 585	0.06610 7621	0.03988 5593	0.02368 4547	0.01387 6041
4	0.24342 253	0.16097 467	0.10358 886	0.06517 2564	0.04023 3041	0.02443 9782
5	0.32254 386	0.22331 414	0.14997 171	0.09818 0383	0.06290 1299	0.03956 0752
6	0.40397 499	0.29182 085	0.20397 470	0.13864 960	0.09202 5082	0.05983 6703
7	0.48402 407	0.36355 664	0.26376 861	0.18575 788	0.12750 123	0.08558 3453
8	0.55979 676	0.43571 936	0.32724 755	0.23825 054	0.16881 131	0.11679 977
9	0.62930 354	0.50590 550	0.39228 077	0.29460 601	0.21509 410	0.15317 249
10	0.69140 835	0.57224 905	0.45691 074	0.35320 331	0.26524 741	0.19411 562
11	0.74568 670	0.63345 596	0.51948 357	0.41246 745	0.31803 966	0.23883 282
12	0.79223 951	0.68876 468	0.57871 406	0.47097 930	0.37221 505	0.28639 161
13	0.83157 044	0.73786 368	0.63369 691	0.52754 591	0.42658 131	0.33579 855
14	0.86433 777	0.78079 126	0.68387 959	0.58123 395	0.48007 442	0.38606 721
15	0.89133 933	0.81783 577	0.72901 217	0.63137 319	0.53192 298	0.43627 343
16	0.91338 014	0.84944 689	0.76908 693	0.67753 837	0.58104 867	0.48559 561
17	0.93122 870	0.87616 330	0.80427 703	0.71951 749	0.62730 382	0.53333 976
18	0.94558 573	0.89855 748	0.83488 047	0.75727 370	0.67022 266	0.57895 097
19	0.95706 933	0.91719 666	0.86127 236	0.79090 589	0.70961 911	0.62201 363
20	0.96621 125	0.93261 711	0.88386 685	0.82061 166	0.74543 806	0.66224 332
21	0.97346 026	0.94530 910	0.90308 831	0.84665 479	0.77772 917	0.69947 312
22	0.97918 933	0.95570 961	0.91935 091	0.86933 815	0.80662 156	0.73363 676
23	0.98370 468	0.96420 055	0.93304 499	0.88898 229	0.83230 099	0.76475 062
24	0.98725 528	0.97111 054	0.94452 908	0.90590 923	0.85499 019	0.79289 604
25	0.99004 194	0.97671 880	0.95412 582	0.92043 089	0.87493 271	0.81820 285

TABLE 3

The factor $2^p(m!)/p^{m+1}$

m	p = 1	p = 2	p = 3	p = 4	p = 5	p = 6
0	2.00000 00	2.00000 00	2.66666 67	4.00000 00	6.40000 00	1.06666 67 (1)
1	2.00000 00	1.00000 00	8.88888 89 (−1)	1.00000 00	1.28000 00	1.77777 78
2	4.00000 00	1.00000 00	5.92592 59 (−1)	5.00000 00 (−1)	5.12000 00 (−1)	5.92592 59 (−1)
3	1.20000 00 (1)	1.50000 00	5.92592 59 (−1)	3.75000 00 (−1)	3.07200 00 (−1)	2.96296 30 (−1)
4	4.80000 00 (1)	3.00000 00	7.90123 46 (−1)	3.75000 00 (−1)	2.45760 00 (−1)	1.97530 86 (−1)
5	2.40000 00 (2)	7.50000 00	1.31687 24	4.68750 00 (−1)	2.45760 00 (−1)	1.64609 05 (−1)
6	1.44000 00 (3)	2.25000 00 (1)	2.63374 49	7.03125 00 (−1)	2.94912 00 (−1)	1.64609 05 (−1)
7	1.00800 00 (4)	7.87500 00 (1)	6.14540 47	1.23046 88	4.12876 80 (−1)	1.92043 90 (−1)
8	8.06400 00 (4)	3.15000 00 (2)	1.63877 46 (1)	2.46093 75	6.60602 88 (−1)	2.56058 53 (−1)
9	7.25760 00 (5)	1.41750 00 (3)	4.91632 37 (1)	5.53710 94	1.18908 52	3.84087 79 (−1)
10	7.25760 00 (6)	7.08750 00 (3)	1.63877 46 (2)	1.38427 73 (1)	2.37817 04	6.40146 32 (−1)
11	7.98336 00 (7)	3.89812 50 (4)	6.00884 01 (2)	3.80676 27 (1)	5.23197 48	1.17360 16
12	9.58003 20 (8)	2.33887 50 (5)	2.40353 60 (3)	1.14202 88 (2)	1.25567 40 (1)	2.34720 32
13	1.24540 42 (10)	1.52026 88 (6)	1.04153 23 (4)	3.71159 36 (2)	3.26475 23 (1)	5.08560 69
14	1.74356 58 (11)	1.06418 81 (7)	4.86048 40 (4)	1.29905 78 (3)	9.14130 64 (1)	1.18664 16 (1)
15	2.61534 87 (12)	7.98141 09 (7)	2.43024 20 (5)	4.87146 66 (3)	2.74239 19 (2)	2.96660 40 (1)
16	4.18455 80 (13)	6.38512 88 (8)	1.29612 91 (6)	1.94858 67 (4)	8.77565 41 (2)	7.91094 40 (1)
17	7.11374 86 (14)	5.42735 94 (9)	7.34473 14 (6)	8.28149 33 (4)	2.98372 24 (3)	2.24143 41 (2)
18	1.28047 47 (16)	4.88462 35 (10)	4.40683 88 (7)	3.72667 20 (5)	1.07414 01 (4)	6.72430 24 (2)
19	2.43290 20 (17)	4.64039 23 (11)	2.79099 79 (8)	1.77016 92 (6)	4.08173 22 (4)	2.12936 24 (3)
20	4.86580 40 (18)	4.64039 23 (12)	1.86066 53 (9)	8.85084 59 (6)	1.63269 29 (5)	7.09787 48 (3)
21	1.02181 88 (20)	4.87241 19 (13)	1.30246 57 (10)	4.64669 41 (7)	6.85731 02 (5)	2.48425 62 (4)
22	2.24800 15 (21)	5.35965 31 (14)	9.55141 51 (10)	2.55568 18 (8)	3.01721 65 (6)	9.10893 93 (4)
23	5.17040 33 (22)	6.16360 11 (15)	7.32275 16 (11)	1.46951 70 (9)	1.38791 96 (7)	3.49176 01 (5)
24	1.24089 68 (24)	7.39632 13 (16)	5.85820 13 (12)	8.81710 21 (9)	6.66201 40 (7)	1.39670 40 (6)
25	3.10224 20 (25)	9.24540 16 (17)	4.88183 44 (13)	5.51068 88 (10)	3.33100 70 (8)	5.81960 01 (6)

m	p = 7	p = 8	p = 9	p = 10	p = 11	p = 12
0	1.82857 14 (1)	3.20000 00 (1)	5.68888 89 (1)	1.02400 00 (2)	1.86181 82 (2)	3.41333 33 (2)
1	2.61224 49	4.00000 00	6.32098 77	1.02400 00 (1)	1.69256 20 (1)	2.84444 44 (1)
2	7.46355 69 (−1)	1.00000 00	1.40466 39	2.04800 00	3.07738 54	4.74074 07
3	3.19866 72 (−1)	3.75000 00 (−1)	4.68221 31 (−1)	6.14400 00 (−1)	8.39286 93 (−1)	1.18518 52
4	1.82780 98 (−1)	1.87500 00 (−1)	2.08098 36 (−1)	2.45760 00 (−1)	3.05195 25 (−1)	3.95061 73 (−1)
5	1.30557 85 (−1)	1.17187 50 (−1)	1.15610 20 (−1)	1.22880 00 (−1)	1.38725 11 (−1)	1.64609 05 (−1)
6	1.11906 72 (−1)	8.78906 25 (−2)	7.70734 66 (−2)	7.37280 00 (−2)	7.56682 43 (−2)	8.23045 27 (−2)
7	1.11906 72 (−1)	7.69042 97 (−2)	5.99460 29 (−2)	5.16096 00 (−2)	4.81525 19 (−2)	4.80109 74 (−2)
8	1.27893 40 (−1)	7.69042 97 (−2)	5.32853 59 (−2)	4.12876 80 (−2)	3.50200 14 (−2)	3.20073 16 (−2)
9	1.64434 37 (−1)	8.65173 34 (−2)	5.32853 59 (−2)	3.71589 12 (−2)	2.86527 38 (−2)	2.40054 87 (−2)
10	2.34906 24 (−1)	1.08146 67 (−1)	5.92059 55 (−2)	3.71589 12 (−2)	2.60479 44 (−2)	2.00045 72 (−2)
11	3.69138 38 (−1)	1.48701 67 (−1)	7.23628 34 (−2)	4.08748 03 (−2)	2.60479 44 (−2)	1.83375 25 (−2)
12	6.32808 66 (−1)	2.23052 50 (−1)	9.64837 78 (−2)	4.90497 64 (−2)	2.84159 39 (−2)	1.83375 25 (−2)
13	1.17521 61	3.62460 32 (−1)	1.39365 46 (−1)	6.37646 93 (−2)	3.35824 73 (−2)	1.98656 52 (−2)
14	2.35043 22	6.34305 55 (−1)	2.16790 71 (−1)	8.92705 70 (−2)	4.27413 30 (−2)	2.31765 94 (−2)
15	5.03664 04	1.18932 29	3.61317 85 (−1)	1.33905 86 (−1)	5.82836 31 (−2)	2.89707 42 (−2)
16	1.15123 21 (1)	2.37864 58	6.42342 85 (−1)	2.14249 37 (−1)	8.47761 91 (−2)	3.86276 56 (−2)
17	2.79584 93 (1)	5.05462 24	1.21331 43	3.64223 93 (−1)	1.31017 75 (−1)	5.47225 13 (−2)
18	7.18932 69 (1)	1.13729 00 (1)	2.42662 85	6.55603 07 (−1)	2.14392 68 (−1)	8.20837 70 (−2)
19	1.95138 87 (2)	2.70106 38 (1)	5.12288 25	1.24564 58	3.70314 63 (−1)	1.29965 97 (−1)
20	5.57539 63 (2)	6.75265 96 (1)	1.13841 83 (1)	2.49129 17	6.73299 33 (−1)	2.16609 95 (−1)
21	1.67261 89 (3)	1.77257 31 (2)	2.65630 94 (1)	5.23171 25	1.28538 96	3.79067 41 (−1)
22	5.25680 23 (3)	4.87457 61 (2)	6.49320 08 (1)	1.15097 67 (1)	2.57077 93	6.94956 92 (−1)
23	1.72723 50 (4)	1.40144 06 (3)	1.65937 35 (2)	2.64724 65 (1)	5.37526 57	1.33200 08
24	5.92194 87 (4)	4.20432 19 (3)	4.42499 61 (2)	6.35339 16 (1)	1.17278 52 (1)	2.66400 15
25	2.11498 17 (5)	1.31385 06 (4)	1.22916 56 (3)	1.58834 79 (2)	2.66542 10 (1)	5.55000 31

N.B. (n) at the end of the number stands for the factor 10^n.

The form of expansion of the function in parentheses into trigonometric series is different according as m and p are even or odd. From the known relations,

$$
(4)\quad
\begin{aligned}
\sin^{2m} x &= \frac{1}{2^{2m}}\left\{\binom{2m}{m} + 2\sum_{k=1}^{m}(-1)^k\binom{2m}{m-k}\cos 2kx\right\},\\
\sin^{2m+1} x &= \frac{1}{2^{2m}}\sum_{k=0}^{m}(-1)^k\binom{2m+1}{m-k}\sin(2k+1)x,
\end{aligned}
$$

the following four forms of expansion are obtained:

$$
(5)\quad
\begin{aligned}
\frac{d^{2p-1}}{dx^{2p-1}}\sin^{2m} x &= \frac{(-1)^p}{2^{2m-2p}}\sum_{k=1}^{m}(-1)^k\binom{2m}{m-k}k^{2p-1}\sin 2kx,\\
\frac{d^{2p}}{dx^{2p}}\sin^{2m} x &= \frac{(-1)^p}{2^{2m-2p-1}}\sum_{k=1}^{m}(-1)^k\binom{2m}{m-k}k^{2p}\cos 2kx,\\
\frac{d^{2p-1}}{dx^{2p-1}}\sin^{2m+1} x &= \frac{(-1)^{p+1}}{2^{2m}}\sum_{k=0}^{m}(-1)^k\binom{2m+1}{m-k}(2k+1)^{2p-1}\cos(2k+1)x,\\
\frac{d^{2p}}{dx^{2p}}\sin^{2m+1} x &= \frac{(-1)^p}{2^{2m}}\sum_{k=0}^{m}(-1)^k\binom{2m+1}{m-k}(2k+1)^{2p}\sin(2k+1)x.
\end{aligned}
$$

By substituting the first and fourth forms into (3) and interchanging the integral and summation signs, the integral is evaluated with the aid of the first of the following two integrals.

$$
(6)\quad
\begin{aligned}
\int_0^\infty \frac{\sin ax}{x}\,dx &= \frac{\pi}{2} \qquad (a > 0),\\
\int_0^\infty \frac{\cos ax - \cos bx}{x}\,dx &= \log\frac{b}{a} \qquad (a, b > 0).
\end{aligned}
$$

However, when the second and third forms are substituted into (3), the integral and summation signs are no longer interchangeable, on account of divergence of the resulting integral. Nevertheless, the divergence can be removed if the following null series are first subtracted from the second and third forms, respectively.

$$
(7)\quad
\begin{aligned}
\frac{(-1)^p}{2^{2m-2p-1}}\sum_{k=1}^{m}(-1)^k\binom{2m}{m-k}k^{2p}\cos 2x &= 0 \qquad (m-1 \geqq p),\\
\frac{(-1)^{p+1}}{2^{2m}}\sum_{k=0}^{m}(-1)^k\binom{2m+1}{m-k}(2k+1)^{2p-1}\cos x &= 0 \qquad (m \geqq p).
\end{aligned}
$$

The signs are then interchangeable. The integral can now be evaluated with the aid of the second integral of (6). The results thus obtained are as follows:

TABLE 4

Values of $S(m, p)$

m	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$
1	1.57079 63	—	—	—	—	—
2	—	1.57079 63	—	—	—	—
3	0.78539 816	0.82395 922	1.17809 72	—	—	—
4	—	0.78539 816	0.69314 718	1.04719 76	—	—
5	0.58904 862	0.52699 967	0.49087 385	0.55069 875	0.94084 155	—
6	—	0.58904 862	0.42175 136	0.39269 908	0.46761 388	0.86393 798
7	0.49087 385	0.41411 904	0.34361 170	0.30702 558	0.31497 739	0.40425 469
8	—	0.49087 385	0.32364 233	0.26179 939	0.24208 057	0.26179 939
9	0.42951 462	0.35144 059	0.27611 654	0.22498 048	0.19788 352	0.19412 115
10	—	0.42951 462	0.27087 430	0.20453 077	0.17003 169	0.15544 339
11	0.38656 316	0.31041 690	0.23623 304	0.18300 322	0.14905 180	0.13022 319
12	—	0.38656 316	0.23708 779	0.17180 585	0.13456 376	0.11290 099
13	0.35434 956	0.28096 110	0.20938 838	0.15710 193	0.12214 322	0.09991 3814
14	—	0.35434 956	0.21321 504	0.15033 012	0.11331 921	0.09019 8070
15	0.32903 888	0.25851 531	0.18983 012	0.13931 449	0.10498 181	0.08228 2234
16	—	0.32903 888	0.19524 517	0.13499 031	0.09907 0524	0.07608 5447
17	0.30847 395	0.24068 795	0.17480 190	0.12622 440	0.09300 3577	0.07073 4477
18	—	0.30847 395	0.18110 988	0.12338 958	0.08878 5022	0.06644 0543
19	0.29133 651	0.22609 027	0.16280 570	0.11611 498	0.08411 6276	0.06256 1822
20	—	0.29133 651	0.16962 534	0.11424 961	0.08096 8150	0.05940 9798
21	0.27676 968	0.21385 433	0.15295 167	0.10802 543	0.07722 5596	0.05645 3793
22	—	0.27676 968	0.16006 061	0.10682 339	0.07479 7532	0.05404 0066
23	0.26418 924	0.20340 652	0.14467 506	0.10137 401	0.07170 2991	0.05170 0962
24	—	0.26418 924	0.15193 762	0.10064 352	0.06978 2683	0.04979 2058
25	0.25318 136	0.19435 071	0.13759 856	0.09578 6685	0.06716 1204	0.04788 6317

m	$p = 7$	$p = 8$	$p = 9$	$p = 10$	$p = 11$	$p = 12$
7	0.80271 510	—	—	—	—	—
8	0.35644 068	0.75298 491	—	—	—	—
9	0.22166 022	0.31863 155	0.71144 659	—	—	—
10	0.15946 942	0.19089 539	0.28808 427	0.67609 865	—	—
11	0.12439 732	0.13324 572	0.16661 695	0.26287 136	0.64553 861	—
12	0.10223 340	0.10144 726	0.11300 633	0.14708 035	0.24171 347	0.61877 683
13	0.08699 5885	0.08165 1983	0.08399 4237	0.09704 3707	0.13108 019	0.22370 496
14	0.07600 1274	0.06831 3278	0.06625 4080	0.07047 2214	0.08423 7253	0.11778 245
15	0.06763 8965	0.05877 1389	0.05447 0870	0.05449 1029	0.05980 9367	0.07380 6336
16	0.06115 1891	0.05165 8629	0.04616 8242	0.04403 5800	0.04535 4545	0.05127 4241
17	0.05590 0617	0.04615 4852	0.04004 5006	0.03676 4798	0.03603 4112	0.03815 0302
18	0.05164 1104	0.04179 5498	0.03537 0732	0.03146 9433	0.02963 7388	0.02980 7803
19	0.04804 1595	0.03824 6985	0.03169 6341	0.02746 9414	0.02503 3476	0.02415 5476
20	0.04503 4947	0.03532 1125	0.02874 3581	0.02435 8878	0.02159 3842	0.02013 4851
21	0.04241 1598	0.03285 3399	0.02632 0632	0.02188 0345	0.01894 5163	0.01716 2925
22	0.04017 5990	0.03076 0348	0.02430 2977	0.01986 5994	0.01685 4376	0.01489 6913
23	0.03817 5653	0.02894 7867	0.02259 5503	0.01819 9973	0.01516 9098	0.01312 4266
24	0.03644 7082	0.02737 8361	0.02113 6195	0.01680 2441	0.01378 6609	0.01170 7477
25	0.03486 8074	0.02599 1170	0.01987 2242	0.01561 4435	0.01263 5004	0.01055 4196

$$
(8)\quad
\begin{aligned}
S(2m, 2p) &= \frac{(-1)^p \pi}{2^{2m-2p+1}(2p-1)!} \sum_{k=1}^{m} (-1)^k \\
&\quad \cdot \binom{2m}{m-k} k^{2p-1} \qquad (m \geqq p \geqq 1), \\
S(2m, 2p+1) &= \frac{(-1)^{p+1}}{2^{2m-2p-1}(2p)!} \sum_{k=2}^{m} (-1)^k \\
&\quad \cdot \binom{2m}{m-k} k^{2p} \log k \qquad (m-1 \geqq p \geqq 1), \\
S(2m+1, 2p) &= \frac{(-1)^p}{2^{2m}(2p-1)!} \sum_{k=1}^{m} (-1)^k \\
&\quad \cdot \binom{2m+1}{m-k} (2k+1)^{2p-1} \log (2k+1) \qquad (m \geqq p \geqq 1), \\
S(2m+1, 2p+1) &= \frac{(-1)^p \pi}{2^{2m+1}(2p)!} \sum_{k=0}^{m} (-1)^k \\
&\quad \cdot \binom{2m+1}{m-k} (2k+1)^{2p} \qquad (m \geqq p \geqq 0),
\end{aligned}
$$

of which the first few values are

$$
(9)\quad
\begin{aligned}
&S(1,1) = \frac{\pi}{2}, \quad S(2,2) = \frac{\pi}{2}, \quad S(3,3) = \frac{3\pi}{8}, \quad S(4,4) = \frac{\pi}{3} \\
&S(3,1) = \frac{\pi}{4}, \quad S(3,2) = \frac{3}{4}\log 3, \quad S(4,3) = \log 2, \quad S(6,4) = \frac{\pi}{8}, \\
&S(5,1) = \frac{3\pi}{16}, \quad S(4,2) = \frac{\pi}{4}, \quad S(5,3) = \frac{5\pi}{32}, \quad S(8,4) = \frac{\pi}{12}.
\end{aligned}
$$

The results rounded to 8D are shown in Table 4. Here the values of natural logarithms are taken from Peters and Stein's *Mathematical Tables* [8]. Naturally, the preceding method of evaluation does not extend to the divergent integral $S(2m, 1)$. The following recurrence relation is found by integration by parts, and may be used for checking purposes.

$$
(10)\quad S(m, p) = \frac{m}{(p-1)(p-2)} \{(m-1)S(m-2, p-2) - mS(m, p-2)\} \qquad (m \geqq p \geqq 3).
$$

It may be mentioned that the particular integral $S(m, m)$ was considered before by several investigators [9]–[13]. Together with a factor $2/\pi$, its values were tabulated to 10D by Harumi, Katsura, and Wrench [12] for $m = 1(1)30$, and by Medhurst and Roberts [13] for $m = 30(1)100$. The last value given by the former is inaccurate, as was pointed out by the latter. It may be added that Grimsey [9] gave these values as rational fractions up to $m = 12$, and Medhurst and Roberts [13] gave four more values, up to $m = 16$.

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Numerical Evaluation of the Elliptic Integral of the Third Kind

By Charles H. Franke

The purpose of this note is to point out a use of the addition formula for the elliptic integral of the third kind which significantly simplifies the numerical evaluation of the function.

The elliptic integral of the third kind may be defined by

$$\Pi(n, k^2, \phi) = \int_0^\phi \frac{d\alpha}{(1 + n \sin^2 \alpha)(1 - k^2 \sin^2 \alpha)^{1/2}}.$$

Two standard power series are used to evaluate $\Pi(n, k^2, \phi)$ [2, p. 5],

$$\text{(1)}\qquad \Pi(n, k^2, \phi) = \sum_{j=0}^{\infty} a(j)A(j)k^{2j}, \qquad a(j) = \frac{(2j)!}{2^{2j}(j!)^2}, \qquad A(j) = \int_0^\phi \frac{\sin^{2j}\alpha}{1 + n \sin^2 \alpha}\,d\alpha,$$

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