On a Constant in the Theory of Trigonometric Series

By Robert F. Church

The note "A constant in the theory of trigonometric series" in the October 1964 issue of *Mathematics of Computation* provided us with a test for our recently constructed algorithms for the computation of roots of functions, and for numerical quadrature in the presence of singularities. The latter algorithm, utilizing the Gaussian 8-point quadrature formula applied to sub-intervals of variable length, involves a sufficiently small number of ordinates that computational labor and round-off error do not become problems. Use of these algorithms indicated the value $\alpha_0 = .3084438$, for the root of the equation $\int_0^{3\pi/2} u^{-\alpha} \cos u \, du = 0$, differing from the reported value, .30483, in the third place. To check this result, we made the transformation $u = x^4$ to weaken the character of the singularity at the origin, and obtained the following table by conventional numerical quadrature, confirming our result:

α	$F(\alpha)$
.308441	$-.99\,(10^{-5})$
.308442	$-.63\,(10^{-5})$
.308443	$-.28\,(10^{-5})$
.308444	$.08\,(10^{-5})$
.308445	$.44\,(10^{-5})$
.308446	$.79\,(10^{-5})$.

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In a recent note, Boas and Klema [1] considered

$$(1) \qquad F(\alpha) = \int_0^{3\pi/2} u^{-\alpha} \cos u \, du, \qquad R(\alpha) < 1,$$

and gave some computations from which they concluded that a zero α_0 of $F(\alpha)$ lies between 0.30483 and 0.30484. Since their tabulated values of $F(\alpha)$ in the vicinity of the root are given to 8D and there are eight such entries, it would seem, since $F(\alpha)$ is analytic for $R(\alpha) < 1$, that the zero could be given to more places by differencing and making use of ordinary inverse interpolation techniques. It is found

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that the differences are not smooth.* This has led us to determine the zero anew. We find that, to 15D,

$$\alpha_0 = 0.30844\ 37795\ 61985.$$

Thus the value in [1] is incorrect in the third place. The computation was done in two different ways using the main diagonal Padé approximations for the incomplete gamma functions $\gamma(\nu, z)$ and $\Gamma(\nu, z)$, see [2, 3, 4]. Thus, with

$$\gamma(\nu, z) = \int_0^z e^{-t} t^{\nu-1}\, dt, \qquad R(\nu) > 0, \tag{2}$$

$$\Gamma(\nu, z) = \int_z^{\infty e^{i\theta}} e^{-t} t^{\nu-1}\, dt \doteq \Gamma(\nu) - \gamma(\nu, z), \tag{3}$$
$$|\theta| < \pi/2, \quad R(\nu) > 0; \qquad |\theta| = \pi/2, \qquad 0 < R(\nu) < 1,$$

we have

$$F(\alpha) = R\{e^{-i\pi\nu/2}\gamma(\nu, ze^{i\pi/2})\}, \quad \nu = 1 - \alpha, \quad z = 3\pi/2. \tag{4}$$

For the evaluation of $\Gamma(\nu)$, we used an (unpublished) expansion in series of Chebyshev polynomials of the first kind. The basic theory for its development can be found in [5]. All calculations were done on an IBM 1620 computer. After locating the value of α_0 to about 8D, we evaluated $F(\alpha)$ for $\alpha = 0.30844\,380 \pm nh, n = 0(1)3$, $h = 0.5 \cdot 10^{-7}$. 20D were carried and the truncation error assured an accuracy of about 18D. The values of $F(\alpha)$ were differenced and α_0 was found by inverse interpolation using the approach outlined in [6]. In this computation third, and higher differences were ignored as the second differences are essentially constant to 18D, and, rounded to 16D, they are $0.105 \cdot 10^{-13}$.

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